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Correlations in quantum plasmas. I. Resummations in Mayer-like diagrammatics

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For a system of point charges that interact through the three-dimensional electrostatic Coulomb potential (without any regularization) and obey the laws of nonrelativistic quantum mechanics with Bose or Fermi statistics, the static correlations are expressed in terms of Mayer-like diagrams. The exchange effects are taken into account systematically and the long-range Coulomb divergencies are exactly resummed in order to get finite diagrams. For this purpose, in the framework of the grand canonical ensemble, the matrix elements of the imaginary-time evolution operator are represented by the Feynman-Kac functional integral according to Ginibre's idea [J. Math. Phys. **6**, 238 (1965); **6**, 252 (1965), **6**, 1432 (1965)] and we exhibit a correspondence between the correlations in the quantum system of point particles and the distribution functions in a *classical* fluid of ''exchange'' loops with Brownian *random* shapes. The size of a loop, which corresponds to the number of particles involved in the corresponding cyclic permutation, and the shape of a loop, which describes the quantum fluctuations, play the part of internal degrees of freedom that must be integrated over when calculating the distributions relative to the quantum point charges. The loops interact through a *two-body* potential that is different from the electrostatic interaction between two charged curves: each line element of a loop interacts only with a discrete number of line elements in every other loop. The linear response theory to an infinitesimal external charge distribution can be written in this formalism by a formula analogous to that of classical statistics, and the loop-fugacity and loop-density expansions of the loop-distribution functions are derived by the usual techniques of Mayer diagrams generalized to extended objects. At large distances r, the loop potential behaves as $1/r$ and, as in the classical case, every Mayer diagram diverges. However, this nonintegrable tail is independent of the shape of the loops and it is *explicitly* and *exactly* resummed by a generalization of the method developed by Meeron [J. Chem. Phys. **28**, 630 (1958); *Plasma Physics* (McGraw-Hill, New York, 1961)], for classical fluids of point entities. Auxiliary $1/r$ bonds are introduced and subdiagrams involving chains of $1/r$ bonds are integrated over first in a systematic way. The new diagrams contain bonds between loops that decay either exponentially or algebraically, with a $1/r^3$ leading term, and the new diagrams are at least conditionally integrable. The part of the quantum particle-particle correlation arising directly from exchange, which is derived from the density of loop, decays faster than any inverse power law, whereas, as shown in the following paper [Phys. Rev. E **53**, 4595 (1996)], the whole quantum particle-particle correlation, which also involves the correlations between loops, decays only as $1/r^6$. [S1063-651X(96)05105-7]

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I. INTRODUCTION

The present series of papers is concerned with the equilibrium static correlations in matter under usual conditions. In this case, strong, weak, and gravitational interactions are negligible [3] and the electrons and nuclei can be seen as point charges that interact through the three-dimensional electrostatic Coulomb force and obey the laws of nonrelativistic quantum mechanics with the adequate (Fermi or Bose) quantum statistics. The correlations between the charges e_α (where α is an index for the n_s various species) are to be determined in the framework of quantum statistical mechanics with the pair potential $e_\alpha e_\gamma v_C(r) = e_\alpha e_\gamma / r$ (with r the distance between the particles). The pure $1/r$ behavior is subtle to be handled at both short and long distances. The short-range singularity would lead to an implosion of the system if all the negative charges, in the present case the electrons, did not obey Fermi statistics [4,5] (see Sec. II). The fast growth of the fermionic kinetic energy when the density increases is needed to balance the attraction between opposite charges. (The same would be true for integrable potentials such as the Yukawa potential.) On the other hand, nonintegrable potentials (that fall off with distance as, or more slowly than, $1/r^3$) usually have no thermodynamic limit. However, the long range of the Coulomb potential does not lead to an explosion of the system.

The harmonicity of the $1/r$ potential is responsible for a very special screening effect, while the thermodynamic limit [6] (see Sec. II) exists for systems in which a local neutrality can be realized in the bulk, though the harmonicity is probably not a necessary condition for the existence of the thermodynamic limit [7]. At equilibrium, the local distribution of particles is sufficiently neutral and isotropic so that, according to Newton's theorem, the electrostatic field vanishes in the bulk. At a macroscopic level, this is exemplified by the local neutrality relation between the densities ρ_α of the various species

$$\sum_\alpha e_\alpha \rho_\alpha = 0. \tag{1.1}$$

This local neutrality also appears if the force decreases faster

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[8] or slower [9] than the Coulomb interaction and if the state is invariant under some translation group. At a microscopic level, the distribution of particles of species α' around a charge e_α is described by the particle-particle correlation $\rho^{(2)T}_{\alpha\alpha'}(\mathbf{r})$ and Coulomb screening means that any charge e_α in the bulk is surrounded by a polarization cloud with a total charge exactly equal to $-e_\alpha$,

$$\int d\mathbf{r} \sum_{\alpha'} e_{\alpha'} \frac{\rho^{(2)T}_{\alpha\alpha'}(\mathbf{r})}{\rho_\alpha} = -e_\alpha . \tag{1.2}$$

Subsequently, the total effective potential created by a charge and its cloud at large distances is no longer the bare $1/r$ Coulomb potential. Moreover, the induced charge density in the presence of an external (classical) infinitesimal charge is exactly related to some charge distribution function through the linear response theory and Eq. (1.2) implies that the total induced charge is finite in a classical as well as in a quantum plasma, in a conductive as well as in a dielectric phase. Moreover, a Coulombic system in dimension 3 is always in a conductive phase, so that the total induced charge around an infinitesimal distribution of charge $\delta q(\mathbf{r})$ exactly compensates the infinitesimal total external charge

$$\int d\mathbf{r} \sum_{\alpha} e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{r}) = -\int d\mathbf{r}\ \delta q(\mathbf{r}). \tag{1.3}$$

According to the linear response theory, (1.3) implies a sum rule for the second moment of the charge-charge correlation function in the classical case [10] (Stillinger-Lovett sum-rule) and for some response-function (inverse static dielectric function), which is indirectly linked to the charge-charge correlation, in the quantum case [11] (see Sec. IV). We notice that the screening rules (1.1)–(1.3) are compatible with a sufficiently fast algebraic decay of the correlations in plasmas in the classical as well as in the quantum case [11].

The point of the present series of papers (hereafter referred to as the present paper and paper II) is to show that the particle-particle correlation function in a multicomponent plasma does have a $1/r^6$ tail, when the statistics is taken into account and the interaction is the pure Coulomb potential (without any regularization). Moreover, the induced charge density in the presence of a localized external infinitesimal charge and the charge-charge correlation are shown to decay as $1/r^8$ and $1/r^{10}$, respectively. A review of the previous works about the precise question of these decays is postponed to the Introduction of paper II. The standard perturbation many-body theory proves not to be very helpful for the investigation of these tails in multicomponent plasmas (as detailed in the following) and in the present paper we build a formalism in which the Feynman-Kac formula [12] is used to write the grand partition function of quantum particles as the *Maxwell-Boltzmann* grand partition function of ''exchange'' loops interacting through a *two-body* potential, in a way slightly different from the formulas obtained by Ginibre [1]. This paper allows one to calculate the quantum correlations in terms of the density of loops, which is directly linked to exchange statistics, and of the correlation between loops, which is induced by interactions. Since the loop distributions can be derived as functional derivatives of their classical grand partition function, diagrammatics that take the exchange effects into account systematically can be built by analogy with the usual Mayer diagrams of classical fluid statistics [13,14]. The diagram divergencies that are associated with the nonintegrability of the Coulomb potential are dealt with by means of an exact global resummation, which is analogous to that performed for classical plasmas by Meeron [15,2]. Some arbitrariness lies in such resummations and the choice made in the present work is aimed to exhibit the exponential screening of charge-charge and multipole-charge interactions and the $1/r^3$ tail of the partially screened multipole-multipole interactions. Indeed, in paper II, the resummed diagrammatics are used to show that the quantum particle-particle correlation decays as $1/r^6$ and to display clearly how the spherical symmetry of the interaction and of the quantum fluctuation distribution for one particle, together with the harmonicity of the Coulomb potential, enforce this power law. Besides, the Mayer-like diagrammatics allow one to exhibit how the exponential classical macroscopic screening makes the power law $1/r^6$ for the particle-particle correlation fall off to $1/r^8$ and $1/r^{10}$ for the induced charge density and the charge-charge correlation, respectively. Ultimately, this formalism provides low-density expansions for weakly degenerated quantum systems and, in particular, for the coefficient of the previous $1/r^6$ tail, as shown in another paper [16].

The program is achieved in the grand canonical ensemble by using the Feynman-Kac formula to represent the matrix elements of the imaginary-time evolution operator in terms of Wiener functional integrals. The complexity due to the noncommutativity of the quantum operators for point particles is replaced by that of performing path integrals, but meanwhile the exponential of the Hamiltonian of the whole system is factored out into a product of scalar exponentials; if the quantum particles interact through two-body forces, each of these scalar exponentials only involves either a two-body potential or the squared distance between two particles that are permuted with each other under a cycle. The quantum system of point particles proves to be equivalent to a classical fluid of ''filaments'' with Brownian random shapes that describe the quantum fluctuations. This point of view is interesting in at least three respects. First, from a technical point of view, the difficulties associated with the noncommutating operators are circumvented and methods of classical statistics can be adapted to the quantum case [1,17–19], as Ginibre did in order to show that low-density expansions of thermodynamic functions are convergent for some class of integrable potentials. The reduced density matrices (quantum analogs of the classical Ursell functions) were expressed in terms of scalar functions and the algebraic formalism introduced by Ruelle in the classical case could be transposed to the quantum system. Second, the Feynman-Kac representation provides helpful insight in mechanisms involving quantum fluctuations [20–22]. For instance, since the extent of a filament is proportional to $\hbar$, the equivalence is convenient to build diagrams that give $\hbar$ expansions for the semiclassical regime in a more systematic way than the usual Wigner-Kirkwood method [20]. Third, the equivalence is useful to disentangle easily the effects of quantum dynamics from those of quantum statistics. A particle that is not exchanged with any other one in a given density-matrix element is associated with a closed filament, whereas p particles that are permuted with one another under a cyclic permutation are described by p open filaments. Then two approaches can be investigated. In the methods used previously for plasmas, the exchange effects were treated perturbatively with respect to a reference system with Boltzmann statistics [23–27]. In the point of view chosen in the present paper, the open filaments are collected into (closed) ''exchange'' loops [1].

The picture that arises from our approach can be summed up as follows. A loop $\mathscr{L}$ is characterized by its position $\mathbf{R}$ (the position of one of the particles involved in the loop) and its internal degrees of freedom. The latter are the species α of the corresponding particles, the size p of the loop (defined as the number of particles exchanged under the corresponding cyclic permutation), and its shape $\mathbf{X}$ (the positions of the other particles and the random Brownian paths that connect them together). The fugacity of a loop contains a contribution that describes the exchange in an ideal gas and a contribution from the self-energy of the loop. Moreover, the loop ''fugacities'' in the present formalism are not necessarily positive; for fermions, the sign of the loop fugacity depends on the number of particles involved in the loop. The self-energy of a loop is positive and the absolute value of the fugacity of interacting loops is lower than for noninteracting loops. (The interaction between identical charges is repulsive and tends to reduce the importance of the exchange effects.) In the potential between loops, each line element of a loop, with a curve abscissa τ, interacts through the Coulomb potential only with the p' line elements of the other loop of size p' whose curve abscissa τ' differs from τ by an integer. (In dimensional units, the corresponding imaginary time runs from 0 to $p\beta\hbar$.) Thus the multipole-multipole part of the forces between the loops is different from the electrostatic interaction and is not exponentially screened *a priori*. The distribution functions for the quantum particles are derived from the distribution functions of the loops by integration over the internal degrees of freedom. The density of the quantum charges is obtained from the density of loops by a summation over the various sizes of loops and an integration over their shapes. The part of the particle-particle correlation that comes from configurations where the two considered particles are exchanged within a cycle appears as a contribution from the loop density integrated over its internal degrees of freedom except two particle positions; the part of the particle-particle correlation built by configurations where the particles are not exchanged together comes directly from the loop correlations. Though the two-body potential between loops with sizes p and p' corresponds in fact to a $(p+p')$-body potential for the corresponding exchanged quantum point particles, the loops obey classical statistics and interact through a *two-body* potential $e_\alpha e_{\alpha'} v(\mathscr{L},\mathscr{L}')$ between objects with an internal structure.

As a consequence, in order to get expansions in powers of the fugacity or of the density of loops, the usual techniques of Mayer diagrams for point entities can be generalized to the system of loops, though the latter is made of objects with various internal degrees of freedom, namely, $(\alpha,p,\mathbf{X})$, which interact through a potential involving not only the distance but also the internal degrees of freedom. Each point of the Mayer diagram is associated with both the position and the internal degrees of freedom of a loop. Two points are linked by at most one bond $f=\exp[-\beta e_\alpha e_{\alpha'} v(\mathscr{L},\mathscr{L}')]-1$, with $\beta=1/k_B T$. In the integral corresponding to a given diagram, each internal point is associated with a measure $\int d\mathscr{L} = \sum_{\alpha=1}^{n_s}\sum_{p=1}^{\infty}\int d\mathbf{R}\int D(\mathbf{X})$ and a weight $z(\mathscr{L})$ $[\rho(\mathscr{L})]$ for the loop-fugacity [loop-density] expansions. The diagrams cannot be calculated explicitly at finite loop fugacity [density] because the weight $z_{\alpha,p}(\mathbf{X})$ $[\rho_{\alpha,p}(\mathbf{X})]$ of each point is a complicated function of the shape $\mathbf{X}$ of the exchange loop, which can be calculated only perturbatively. At large distances r, the potential between the loops behaves as the $1/r$ interaction between the total charges of the loops (the total charge of a loop is the sum of the charges of the corresponding particles). As in the classical case, every Mayer graph diverges because of the nonintegrable long range of the Coulomb potential. By taking advantage of the fact that the $1/r$ asymptotic behavior of the potential does not depend on the shapes of the loops, these tails are *explicitly* and *exactly* resummed by a generalization of the method developed by Meeron. The auxiliary bonds (which are introduced in the process in a partially arbitrary way) are chosen to exhibit the decomposition of the loop potential into charge-charge, multipole-charge, and multipole-multipole interactions. The subdiagrams involving chains of $1/r$ bonds are integrated over first. The corresponding collective effect ensures that the charge-charge and charge-multipole interactions are exponentially screened, as in the classical case, with a screening length that coincides with the classical Debye-Hückel value when the exchange effects become negligible. However, the multipole-multipole interaction is only partially screened and the corresponding Mayer bond decays algebraically at large distances, with a dipole-dipole-like leading term. Subsequently, the loop correlation decays as $1/r^3$, while the loop density falls off faster than any inverse power law when the distance between the positions of two particles involved in the loop becomes infinite because of the part of the loop fugacity that comes from the exchange in vacuum. Eventually, the part of the quantum particle-particle correlation that emerges directly from exchange decays faster than any inverse power law (except in a phase analogous to a Bose condensation, where the correlation tends to a finite constant value plus fast-decaying corrections). As shown in paper II, the part of the particle-particle correlation arising from configurations where the particles are not exchanged together under the same cyclic permutation behaves as $1/r^6$ at large distances.

The existence of the $1/r^3$ dipole-dipole-like interaction between the charges surrounded by their polarization clouds can be traced back in other formalisms that go beyond the mean-field approach. This interaction appears in some intermediate quantities that we call ''chain potentials,'' because they are obtained by summing chains of auxiliary bonds with the aim of taking into account the collective effects that partially screen the large-distance Coulombic tails of the corresponding bare potential. For instance, in the standard many-body theory, the chain potential is the random-phase approximation effective potential. Other examples in the approximation of Maxwell-Boltzmann statistics can be found in Refs. [21, 25] and its classical analog is the Debye-Hückel potential. At finite temperature, the resulting chain potential has a short-ranged part and a long-ranged part that decays algebraically. The first model of quantum plasma to be investigated was the one-component plasma (OCP), also called the ''jellium''; it is a system made up with identical point particles that move in a *continuous* uniform fixed background of the opposite charge that ensures the local neutrality (1.1). In the particular case of the OCP, where the particle and charge densities are proportional to each other, two subsequent exact sum rules allow one to show that an algebraic falloff of the basic graphs of the standard formalism (proper polarization graphs) might lead to an algebraic falloff of the particle-particle correlation (if compensations do not occur). However, in the case of multicomponent plasmas, these rules do not hold and one cannot iterate the arguments that can be developed for the OCP. Eventually, the point of view of the Feynman-Kac representation proves to be the most adequate one for the investigation of the large-distance behaviors of the correlations and induced charge density in a multicomponent plasma. In particular, it allows one to study the coefficients of the algebraic tails at low density, as shown in another paper [16].

The paper is organized as follows. In Sec. II A we recall known results about the stability of Coulomb systems and the existence of the thermodynamic limit of the quantum grand partition function Ξ. The distribution functions are introduced in Sec. II B. In Appendix A we show how, for a Hamiltonian independent from the spin, the notion of ''exchange loops'' emerges from the fact that any permutation can be expressed uniquely as a product of cycles with no common elements. The Feynman-Kac formula then leads straightforwardly to the equality between Ξ and the grand partition function of a classical gas of exchange loops interacting via a two-body potential (Sec. III A). In Sec. III B we give a compact formulation of the equivalence, which is particular to the present work. A multipolar decomposition of the loop potential (Sec. III C) shows that the difference between the loop potential and the electrostatic potential between charged loops appears only in the multipole-multipole part of the interaction. (This remark is valid for any potential.) In Sec. III D the distribution functions of the classical loops are rewritten in terms of functional derivatives of the grand partition Ξ with respect to the fugacities of the loops, by means of formulas analogous to those encountered in classical statistics. The correspondence between the quantum system of point particles and the classical system of loops allows one to relate the distribution functions of the quantum particles to those of the exchange loops (Sec. IV A). These general formulas are checked in the noninteracting case (Sec. IV B). The theory of linear response to an external static charge distribution is displayed in Sec. IV C: the induced charge is written in terms of the loop distributions and then in terms of the charge-charge correlation function. In Sec. IV D two sum rules describing Coulombic screening are derived in the present formalism. The finiteness of the total induced charge allows one to retrieve the zero-moment sum rule that is obeyed by the charge-charge correlation according to (1.2); moreover, it implies the positiveness of an expression that involves the zero moment of the part of the correlation directly induced by quantum statistics. The perfect screening sum rule (1.3) specific to a conducting phase is given in terms of the second moment of the charge-charge correlation plus a quantum correction involving the loop distributions. In Sec. V we introduce the virial diagrams for the system of loops, the Coulomb divergencies are resummed, and the part of the correlation arising from exchange statistics is shown to decay faster than any inverse power law. The technical lines of Sec. V are summed up at the beginning of the section. As a conclusion, the exchange loop formalism is compared with the standard many-body perturbation formalism in Sec. VI. In Sec. VI A the general structures of both diagrammatics are compared and the linear response that gives the indirect relation between the induced charge density and the quantum particle-particle correlation is recalled for a multicomponent plasma in the standard many-body perturbation theory. In both formalisms, the diagram divergencies that are linked to the large-distance nonintegrability of the Coulomb potential are dealt with by an exact partial resummation of subdiagrams that are kinds of interaction-line chains (Sec. VI B). The corresponding chain potential involves an algebraic part, but the consequence of the large-distance behavior of the quantum particle-particle correlation can be investigated in the standard perturbation theory only in the case of the one-component plasma (Sec. VI C).

II. GRAND CANONICAL ENSEMBLE

A. Stability and thermodynamic limit

In this series of papers we consider systems made of n_s species α of point charges ($\alpha=1,...,n_s$) in an infinite volume. Let N_α be the number of charges of species α. The particle with index i ($i=1,\ldots,\Sigma_\alpha N_\alpha$) has a charge e_{α_i}, a mass m_{α_i}, and a spin number S_{α_i}. Its quantum state is characterized by its position $\mathbf{r}_i$ and the projection $\hbar S^z_{\alpha_i}(i)$ of its spin $\hbar \mathbf{S}_{\alpha_i}$ along a given axis z ($S^z_{\alpha_i}$ can take the $2S_{\alpha_i}+1$ values $-S_{\alpha_i}$, $-S_{\alpha_i}+1,...,S_{\alpha_i}$ and $\hbar$ is the Planck constant). The charges interact through the Coulomb pair potential $e_{\alpha_i}e_{\alpha_j}v_C(r_{ij})=e_{\alpha_i}e_{\alpha_j}/r_{ij}$ (with $r_{ij}=|\mathbf{r}_{ij}|$ and $\mathbf{r}_{ij}=\mathbf{r}_j-\mathbf{r}_i$) and their quantum Hamiltonian operator $\hat{H}_{\{N_\alpha\}}$ in position representation reads

$$\hat{H}_{\{N_\alpha\}}=\sum_i \left(\frac{-\hbar^2}{2m_{\alpha_i}}\right)\Delta_{\mathbf{r}_i}+\frac{1}{2}\sum_{i\neq j}\frac{e_{\alpha_i}e_{\alpha_j}}{r_{ij}}. \tag{2.1}$$

First, we recall some results about the canonical ensemble.

The stability of matter with respect to an implosion that would be induced by the strong $1/r$ attraction between charges with opposite signs is a subtle question. For a system with a finite number of particles, the Hamiltonian (2.1) has a finite lower bound [28] (and, in particular, the atoms are stable). The uncertainty principle arising from quantum dynamics ensures that a localized particle has a large kinetic energy that balances the attractive potential energy and this prevents any finite set of charges from collapsing together. Henceforth, at the inverse temperature $\beta=1/k_BT$, the canonical partition function of the system confined in a box of finite volume Λ,

$$Q(\beta,\{N_\alpha\},\Lambda)=\mathrm{Tr}_\Lambda\, e^{-\beta\hat{H}_{\{N_\alpha\}}}, \tag{2.2}$$

is finite. In (2.2) the trace Tr is taken over a basis of quantum states that are symmetric (antisymmetric) with respect to the permutations of the N_α particles of the same bosonic (fermionic) species α. The wave functions vanish on the wall of the box (Dirichlet boundary conditions) and the trace Tr depends on the shape and on the size Λ of the box. Moreover, if there is no collapse in the thermodynamic limit (stability of bulk matter), the energy per particle must have a lower bound and this bound must be independent of the positions of the particles and of the size $\Sigma_\alpha N_\alpha$ of the system. This is the so-called H-stability condition $H_{\{N_\alpha\}} > -B\Sigma_\alpha N_\alpha$. Dyson and Lenard [4,5] showed that, thanks to the Pauli principle, the fermionic kinetic energy increases with the density sufficiently fast so as to compensate the Coulomb attraction energy between opposite charges and the H stability requires all species with negative and/or positive charges to be fermions. This condition is met in real matter where all the negative charges (electrons) are fermions. The density of free energy $F_\Lambda(\beta,\{\rho_\alpha\}) = (1/\beta\Lambda)\ln Q$ per unit volume may have an intensive thermodynamic limit.

On the other hand, if the global net charge is not too large, the system does not explode, in spite of the long-ranged repulsion between charges with opposite signs, and the Gibbs formalism has a proper thermodynamic limit. This was remarkably shown by Lieb and Lebowitz [6] via an argument that exhibits the fundamental physical mechanisms. When a domain is packed with spheres, the densities inside the spheres are radial because of the rotational invariance of the potential. According to Newton's theorem, outside an isotropic distribution of charge, all the charge appears to be concentrated at the center. So, despite the long-ranged nature of the Coulomb potential, the various neutral parts of a system far away from each other are approximately independent and, by taking the thermodynamic limit of the Gibbs canonical partition function, one gets an extensive energy and an extensive free energy together with the intensive thermodynamic variables, such as the pressure. Moreover, if the system is overall neutral, the free energy does not depend on the shape of the domains used in the process that leads to the infinite volume limit, as it is the case for potentials that are short ranged. However, the long-ranged nature of the Coulomb forces has manifest consequences if the global neutrality is not satisfied. In the canonical ensemble, if the excess charge is both ''non-negligible'' and ''not too large,'' it goes to a thin layer near the surface of the system and the density of free energy per unit volume is the sum of the free-energy density of a neutral system plus a term that involves the shape-dependent electrostatic capacity. If the net charge is too large, the density of free energy explodes in the infinite volume limit.

The latter property is linked to the equivalence between the various statistical ensembles [6], which is quite particular in the case of Coulomb interactions. The thermodynamic properties (or intensive variables) are the same whether they are given by the grand canonical ensemble (which involves nonneutral systems) or by the canonical or microcanonical ensembles for neutral systems. Indeed, let us consider the quantum grand partition function of the system when a chemical potential μ_α is associated with each species α,

$$\Xi(\beta,\{\mu_\alpha\},\Lambda) = \sum_{\{N_\alpha\}_{\alpha=1,\ldots,n_s}} \mathrm{Tr}_\Lambda\left\{\exp\left[-\beta\left(\hat{H}_{\{N_\alpha\}} - \sum_\alpha \mu_\alpha \hat{N}_\alpha\right)\right]\right\}, \tag{2.3}$$

where N_α runs from 0 to ∞. Regardless of the choice of the chemical potentials μ_α of the various species, the non-neutral systems in (2.3) make a vanishingly small contribution to the grand canonical pressure and the densities of the various species that appear in the thermodynamic limit obey the neutrality requirement (1.1). Moreover, in the grand canonical ensemble, a finite density of charge in the infinite volume limit cannot appear because the self-energy of the corresponding surface charge becomes infinite and its contribution to Ξ is exponentially small.

We mention also the results about the OCP, where there is only one species of moving charges immersed in a continuous uniform fixed background of the opposite sign. [Its classical version is a model for classical ions moving in the rigid bath made up by degenerate quantum electrons [29]. The quantum OCP is often used as a first approximation for the description of the conduction electron fluid in a metal when the Fermi surface is nearly spherical (for instance, in alkali metals) and when the electron-phonon interactions are not relevant for the phenomena to be studied [30].] In the case of the OCP, the Hamiltonian involves the self-energy of the background and the interaction of the background with the moving charges. The H stability holds in the classical sense and Fermi statistics is not required [31]. The neutral and non-neutral canonical ensembles have well-defined thermodynamic limits, as in the case of multicomponent plasmas. The grand partition function in which the background density ρ_B is fixed (so that the system is not neutral, except for one value of N) exists. (The grand partition function in which the system is neutral for every N diverges in the classical case and in the quantum bosonic case, even for a finite volume, and is finite only in the quantum fermionic case, where the kinetic energy dominates the electrostatic contribution.) In the infinite volume limit, the density in the bulk is equal to the density of the background.

According to the general formalism of statistical mechanics, the average value of an operator $\hat{O}$ in the Fock space is given by

$$\langle \hat{O} \rangle_{\Xi_\Lambda} = \frac{1}{\Xi(\beta,\{\mu_\alpha\},\Lambda)} \sum_{\{N_\alpha\}_{\alpha=1,\ldots,n_s}} \mathrm{Tr}_\Lambda\left\{\hat{O}_{\{N_\alpha\}} \times \exp\left[-\beta\left(\hat{H}_{\{N_\alpha\}} - \sum_\alpha \mu_\alpha \hat{N}_\alpha\right)\right]\right\}. \tag{2.4}$$

In the limit of an infinite volume, the thermodynamic functions in the bulk become independent from the boundary terms arising from the interactions with the walls. Subsequently, when considering the average values (2.4), we can take states that extend over an infinite volume from the beginning of the calculations.

B. Distribution functions

The local density $\rho^Q_{\alpha,\Lambda}(\mathbf{r})$ of the quantum particles of species α is the average value of the operator

$$\hat{\rho}_\alpha(\mathbf{r}) = \sum_i \delta_{\alpha_i,\alpha}\delta(\mathbf{r}_i - \mathbf{r}). \tag{2.5}$$

In the same way, the quantum truncated two-body distribution function $\rho^{(2)TQ}_{\alpha_a\alpha_b,\Lambda}(\mathbf{r}_a,\mathbf{r}_b)$ for species α_a and α_b is defined as

$$\rho^{(2)TQ}_{\alpha_a\alpha_b,\Lambda}(\mathbf{r}_a,\mathbf{r}_b)=\left\langle \sum_{i\neq j}\delta_{\alpha_i,\alpha_a}\delta_{\alpha_j,\alpha_b}\delta(\mathbf{r}_i-\mathbf{r}_a)\delta(\mathbf{r}_j-\mathbf{r}_b)\right\rangle_{\Xi_\Lambda} -\rho^Q_{\alpha_a,\Lambda}(\mathbf{r}_a)\rho^Q_{\alpha_b,\Lambda}(\mathbf{r}_b). \tag{2.6}$$

In the following, we use the superscript Q systematically in order to distinguish quantities calculated with the quantum statistics from averages obtained with the Maxwell-Boltzmann statistics. For a finite system, according to (2.4), and since $\hat{N}_\alpha=\int_\Lambda d\mathbf{r}'\hat{\rho}_\alpha(\mathbf{r}')$, one gets the same relation as in classical statistics

$$\left.\frac{\partial\rho^Q_{\alpha,\Lambda}(\mathbf{r})}{\partial(\beta\mu_\alpha)}\right|_{\beta,\Lambda}=\int_\Lambda d\mathbf{r}'\langle\hat{\rho}_\alpha(\mathbf{r})\hat{\rho}_\alpha(\mathbf{r}')\rangle_{\Xi_\Lambda} -\langle\hat{\rho}_\alpha(\mathbf{r})\rangle_{\Xi_\Lambda}\int_\Lambda d\mathbf{r}'\langle\hat{\rho}_\alpha(\mathbf{r}')\rangle_{\Xi_\Lambda} =\rho^Q_{\alpha,\Lambda}(\mathbf{r})+\int_\Lambda d\mathbf{r}'\rho^{(2)TQ}_{\alpha\alpha,\Lambda}(\mathbf{r},\mathbf{r}'). \tag{2.7}$$

There is no rigorous result about the existence of the thermodynamic limit of the correlations for Coulomb potential. However, the following arguments of the paper might give some hints for a mathematical proof. Moreover, for a quantum two-component plasma of distinguishable and symmetric opposite charges interacting through a Coulomb potential regularized at the origin, the thermodynamic limit of the correlations in the grand canonical ensemble exists at arbitrary temperatures and chemical potentials [32]. We assume that, in the case of a generic multicomponent plasma with quantum statistics, the thermodynamic limit still exists and is independent from boundary effects.

The translational invariance of the Coulomb potential implies that, in the thermodynamic limit and in the bulk, $\rho^Q_\alpha(\mathbf{r})$ is independent from $\mathbf{r}$, while $\rho^{(2)TQ}_{\alpha_a\alpha_b}(\mathbf{r}_a,\mathbf{r}_b)$ depends on $\mathbf{r}_a$ and $\mathbf{r}_b$ only through the distance $r_{ab}=|\mathbf{r}_a-\mathbf{r}_b|$. The uniform bulk density ρ^Q_α can be calculated as $\rho^Q_\alpha=\lim_{\Lambda\to\infty}\langle N_\alpha/\Lambda\rangle_{\Xi_\Lambda}$ and can be derived directly from the infinite volume limit of $(1/\Lambda)\ln\Xi_\Lambda$,

$$\rho^Q_\alpha=\lim_{\Lambda\to\infty}\frac{1}{\Lambda}\frac{\partial(\ln\Xi_\Lambda)}{\partial(\beta\mu_\alpha)}=\frac{\partial P}{\partial\mu_\alpha}, \tag{2.8}$$

where

$$P=\lim_{\Lambda\to\infty}\frac{\ln\Xi_\Lambda}{\beta\Lambda} \tag{2.9}$$

is the thermodynamic limit of the grand canonical bulk pressure P. In a multicomponent plasma, the grand canonical pressure coincides with the kinetic pressure that measures the transfer of particle momentum to the wall. However, in the case of the OCP, the Hamiltonian $\hat{H}_{N,\rho_B}$ involves the background and the grand canonical pressure is not equal to the kinetic pressure because it contains an extra electrostatic contribution associated with the work necessary to deform the background [33]. (The classical grand canonical pressure becomes negative at low temperature, whereas the kinetic pressure remains positive.) In the thermodynamic limit, (2.7) reads

$$\left.\frac{\partial\rho^Q_\alpha}{\partial(\beta\mu_\alpha)}\right|_\beta=\rho^Q_\alpha+\int d\mathbf{r}\;\rho^{(2)TQ}_{\alpha\alpha}(\mathbf{r}). \tag{2.10}$$

For a multicomponent plasma, (2.10) is the usual Ornstein-Zernicke relation. In the OCP, where there is only one species of moving particles, (1.2) reads $\int d\mathbf{r}\;\rho^{(2)TQ}(\mathbf{r})=-\rho^Q$ and (2.10) implies that $\partial\rho^Q/\partial(\beta\mu)=0$. The latter equations reflect the fact that, in the thermodynamic limit, the fluctuations of bulk density as well as the bulk density itself are determined only by the density of the background and do not depend on the chemical potential that is involved in the nonneutral grand partition function. [We recall that, in the same way, the bulk densities in a multicomponent plasma are linked by the local neutrality relation (1.1) and only n_s-1 chemical potentials μ_α are relevant in the thermodynamic limit.] The chemical potential μ^* that appears in the usual thermodynamic relations is another chemical potential, defined as the variation of the free energy when one particle is added to the neutral system, while the latter is kept neutral by a corresponding change of the total charge of the background. For instance, $\partial\rho^Q/\partial\mu^*$ is related to the isothermal compressibility χ_T that measures the response of the density to a variation of the ''thermal'' pressure P_θ, $\chi_T=(1/\rho^Q)(\partial\rho^Q/\partial P_\theta)|_T$, by the usual thermodynamic relation

$$\frac{\partial\rho^Q}{\partial\mu^*}=[\rho^Q]^2\chi_T. \tag{2.11}$$

The thermal pressure P_θ is defined as the opposite of the variation of the free energy per unit volume when the background is compressed together with the moving particles, so that the net charge of the system is kept constant [33]. It is equal to the pressure (2.9) derived in the nonneutral grand canonical ensemble with a chemical potential μ^*, $P_\theta=\lim_{\Lambda\to\infty}(1/\beta\Lambda)\ln\Xi(\beta,\mu^*(\rho_B),\rho_B,\Lambda)$.

III. CLASSICAL GAS OF EXCHANGE LOOPS

A. Feynman-Kac representation

In Appendix A we show how the grand partition function Ξ (2.3) can be reexpressed in terms of the classes of permutations with the same irreducible cyclic structure. Ξ is written in a basis of properly symmetrized tensorial products of individual particle states and the permutations are expressed as products of cycles with no common elements. Since the permutations that have the same cyclic structure are conjugated to one another by a permutation and since the Hamiltonian is invariant under permutations of the indices of the particles of the same species, the grand partition function of a system where the interactions are independent of the spin can eventually be written as

$$\Xi=\sum_{\{n_p^\alpha\}_{p=1,\ldots,\infty}^{\alpha=1,\ldots,n_s}}\prod_{n_p^\alpha\neq 0}\frac{1}{n_p^a!}\left(\frac{\eta_\alpha^{p-1}(2S_\alpha+1)e^{p\beta\mu_\alpha}}{p}\right)^{n_p^\alpha}$$
$$\times\int\prod_i d\mathbf{r}_i\langle\{\mathbf{r}_{\pi^0(i)}\}|e^{-\beta\hat{H}_{\{n_p^\alpha\}}}|\{\mathbf{r}_i\}\rangle, \tag{3.1}$$

where $|\{\mathbf{r}_i\}\rangle\equiv\otimes_i|\mathbf{r}_i\rangle$, π^0 is a composition of particular permutations π_α^0 characterized by the sequence $\{n_p^\alpha\}$ in which n_p^α is the number of cycles involving p particles of species α in the cyclic decomposition of π_α^0 and i ranges from 1 to $\Sigma_\alpha\Sigma_p p n_p^\alpha$. η_α is equal to 1 for bosons and to -1 for fermions. The combinatorics is identical to that found by Ginibre for the reduced density matrices on p. 360 of Ref. [17]. A formula analogous to (3.1), without the spin degeneracy factor, was also retrieved by Hǿye and Stell [19].

The noncommutativity between the kinetic and interaction quantum operators does not allow one to reexpress $\exp(-\beta\hat{H}_{\{n_p^\alpha\}})$ as a product of exponentials, each of which would involve the Hamiltonian of the particles that are permuted within a cycle of π^0. However, according to the well-known Feynman-Kac formula [12,34], every matrix element in Ξ can be written in terms of the path integral of the exponential of i times a classical action with imaginary times $t=-i\beta\hbar s$. In the path integral, the kinetic terms are disentangled from the interaction energies as

$$\langle\{\mathbf{r}_{\pi(i)}\}|e^{-\beta\hat{H}_{\{n_p^\alpha\}}}|\{\mathbf{r}_i\}\rangle$$
$$=\prod_i\left[\frac{1}{(2\pi\lambda_{\alpha_i}^2)^{3/2}}e^{-(\mathbf{r}_{\pi(i)}-\mathbf{r}_i)^2/2\lambda_{\alpha_i}^2}\right]$$
$$\times\int\left[\prod_i D(\boldsymbol{\xi}_i)\right]\exp\left(-\beta\frac{1}{2}\sum_{i\neq j}e_{\alpha_i}e_{\alpha_j}\right.$$
$$\left.\times\int_0^1 ds\, v_C[\boldsymbol{\omega}_{i,\pi(i)}(s)-\boldsymbol{\omega}_{j,\pi(j)}(s)]\right). \tag{3.2}$$

$\boldsymbol{\omega}_{i,\pi(i)}(s)$ is a Brownian path starting from $\mathbf{r}_i$ at time $s=0$ and ending at $\mathbf{r}_{\pi(i)}$ at $s=1$,

$$\boldsymbol{\omega}_{i,\pi(i)}(s)=(1-s)\mathbf{r}_i+s\mathbf{r}_{\pi(i)}+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s), \tag{3.3}$$

where λ_{α_i} is the thermal de Broglie wavelength defined as $\lambda_{\alpha_i}=(\beta\hbar^2/m_{\alpha_i})^{1/2}$ and $\boldsymbol{\xi}_i(s)$ is a Brownian bridge that vanishes when $s=0$ and 1. Its measure $D(\boldsymbol{\xi})$, which contains the exponential of the kinetic part of the Euclidean action, has a Gaussian covariance

$$\int D(\boldsymbol{\xi})[\boldsymbol{\xi}(s)]_\mu[\boldsymbol{\xi}(s')]_\nu=\delta_{\mu,\nu}\mathrm{inf}(s,s')[1-\mathrm{sup}(s,s')], \tag{3.4}$$

where $\mathrm{inf}(s,s')$ $[\mathrm{sup}(s,s')]$ denotes the infimum [supremum] of s and s'.

For a given permutation π^0, we make a partition of the variables $\{\mathbf{r}_i\}$ and order them in such a way that the new variable $\mathbf{x}_l^{\alpha,p,k}$ denotes the position of a particle of species α that is permuted under π^0 inside the kth cycle of length p $(k=1,\ldots,n_p^\alpha)$, with l changed into $l+1$, for $l=1,\ldots,p-1$ and $l=p$ changed into $l=1$ under π^0. Let $\boldsymbol{\omega}_l^{\alpha,p,k}$ be the Brownian path linking $\mathbf{x}_l^{\alpha,p,k}$ to $\mathbf{x}_{l+1}^{\alpha,p,k}$, and $\boldsymbol{\xi}_l^{\alpha,p,k}$ the corresponding Brownian bridge. The so-called kth ''loop'' of species α and length p is the object $\mathscr{L}_k^{\alpha,p}=(\alpha,p,k,\{\mathbf{x}_l^{\alpha,p,k}\}_{l=1,\ldots,p},\ \{\boldsymbol{\xi}_l^{\alpha,p,k}\}_{l=1,\ldots,p})$ (see Fig. 1) and we use the notation $\mathscr{D}(\mathscr{L}_k^{\alpha,p})\equiv\Pi_{l=1}^p[d\mathbf{x}_l^{\alpha,p,k}D(\boldsymbol{\xi}_l^{\alpha,p,k})]$. With every loop we can associate a fugacity

$$z(\mathscr{L}_k^{\alpha,p})=z_{\alpha,p}^*e^{-\beta E_\beta^{\mathrm{int}}(\mathscr{L}_k^{\alpha,p})}, \tag{3.5}$$

where

$$z_{\alpha,p}^*=\eta_\alpha^{p-1}(2S_\alpha+1)\frac{1}{p}\left(\frac{e^{\beta\mu_\alpha}}{(2\pi\lambda_\alpha^2)^{3/2}}\right)^p. \tag{3.6}$$

E_β^{int} is an internal energy defined as

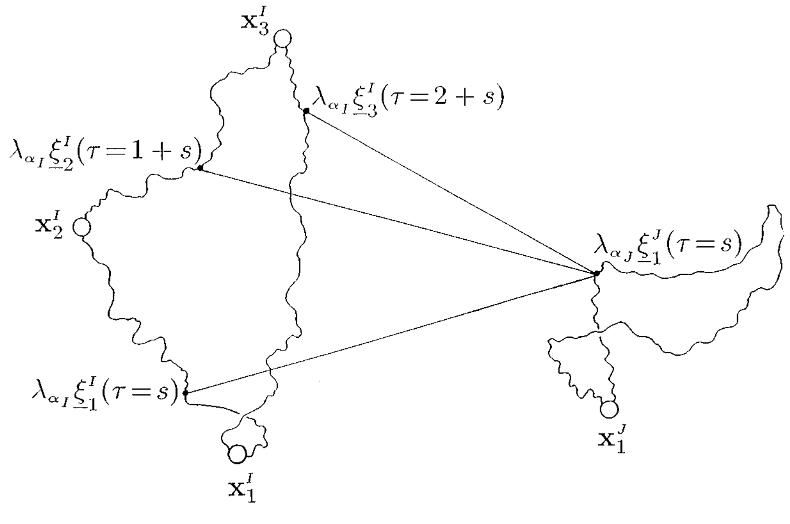

FIG. 1. Potential between two exchange loops $\mathscr{L}_I=(\alpha_I,p_I=3;\ \mathbf{R}_I=\mathbf{x}_1^I;\ \{\mathbf{t}_1^I=\mathbf{x}_2^I-\mathbf{x}_1^I,\ \mathbf{t}_2^I=\mathbf{x}_3^I-\mathbf{x}_2^I\},\ \{\boldsymbol{\xi}_1^I,\boldsymbol{\xi}_2^I,\boldsymbol{\xi}_3^I\})$ and $\mathscr{L}_J=(\alpha_J,p_J=1,\mathbf{R}_J=\mathbf{x}_1^J,\boldsymbol{\xi}_1^J)$, defined by the integral over τ and τ' (3.13). (A bold $\boldsymbol{\xi}$ represents a vector $\boldsymbol{\xi}$ and, for conciseness, we have denoted a point $\mathbf{x}_i+\lambda_{\alpha i}\boldsymbol{\xi}_i$ in a loop only by $\lambda_{\alpha i}$.) The contribution from a given $\tau'=s$, with $0\leqslant s\leqslant 1$, reads

$$e_{\alpha_I}e_{\alpha_J}\int_0^{p_I}d\tau\,\delta([\tau-P(\tau)]$$
$$-[\tau'-P(\tau')])v_C(\boldsymbol{\Omega}_I(\tau)-\boldsymbol{\Omega}_J(\tau'))$$
$$=e_{\alpha_I}e_{\alpha_J}\sum_{l=1}^{3}v(\boldsymbol{\Omega}_I(l+s)-\boldsymbol{\Omega}_J(s))$$

and is represented by three solid lines. The shape $\boldsymbol{\xi}$ made with Brownian paths may have some erratic structure and the Brownian paths may cross themselves or each other. (However, in dimension 3, such paths have a vanishing measure.)

$$E_\beta^{\mathrm{int}}(\mathscr{L}_k^{\alpha,p})=\frac{1}{\beta 2\lambda_\alpha^2}\sum_{l=1}^{p}(\mathbf{x}_{l+1}^{\alpha,p,k}-\mathbf{x}_l^{\alpha,p,k})^2+\frac{e_\alpha^2}{2}\sum_{l\neq l'}\int_0^1 ds\, v_C[\boldsymbol{\omega}_l^{\alpha,p,k}(s)-\boldsymbol{\omega}_{l'}^{\alpha,p,k}(s)],\tag{3.7}$$

with the convention $\mathbf{x}_{p+1}\equiv\mathbf{x}_1$. The interaction potential between the loops is

$$v(\mathscr{L}_k^{\alpha,p},\mathscr{L}_{k'}^{\alpha',p'})=\sum_{l=1}^{p}\sum_{l'=1}^{p'}\int_0^1 ds\, v_C[\boldsymbol{\omega}_l^{\alpha,p,k}(s)-\boldsymbol{\omega}_{l'}^{\alpha',p',k'}(s)].\tag{3.8}$$

With these notations, (3.1) can be written as the grand partition function of a classical gas of objects $\mathscr{L}_k^{\alpha,p}$ interacting through the potential (3.8),

$$\Xi=\Xi_{\mathrm{loop}}\equiv\sum_{\{n_p^\alpha\}_{p=1,\dots,\infty}^{\alpha=1,\dots,n_s}}\prod_{n_p^\alpha\neq 0}\frac{1}{n_p^\alpha!}\int\prod_{(\alpha,p)/n_p^\alpha\neq 0}\times\prod_{k=1}^{n_p^\alpha}[z(\mathscr{L}_k^{\alpha,p})\mathscr{D}(\mathscr{L}_k^{\alpha,p})]\times\exp\left(-\beta\frac{1}{2}\sum_{\alpha,\alpha'}e_\alpha e_{\alpha'}\sum_{p,p'}\sum_{k,k'}{}^{*}\,v(\mathscr{L}_k^{\alpha,p},\mathscr{L}_{k'}^{\alpha',p'})\right),\tag{3.9}$$

where $\Sigma_{k,k'}^{*}$ means that if $\alpha=\alpha'$ and $p=p'$, then $k\neq k'$.

B. Compact formulation

Ξ can be written in an even more compact form by labeling a loop with only one index I ($I=1,\dots,N$ with $N=\Sigma_{\alpha=1}^{n_s}\Sigma_{p=1}^{\infty}n_p^\alpha$) and by using the notation $\mathscr{L}_I=\mathscr{L}_{k_I}^{\alpha_I,p_I}$. According to the identity

$$\sum_{\{n_p^\alpha\}_{p=1,\dots,\infty}^{\alpha=1,\dots,n_s}}\prod_{n_p^\alpha\neq 0}\frac{1}{n_p^\alpha!}\int\prod_{I=1}^{\Sigma_{\alpha,p}n_p^\alpha}D(\mathscr{L}_{k_I}^{\alpha_I,p_I})\cdots=\sum_{N=0}^{\infty}\frac{1}{N!}\int\prod_{n=1}^{N}d\mathscr{L}_n\cdots,\tag{3.10}$$

with the notation $\int d\mathscr{L}_n=\Sigma_{\alpha_n=1}^{n_s}\Sigma_{p_n=1}^{\infty}\int\mathscr{D}(\mathscr{L}_{k_n}^{\alpha_n,p_n})$, (3.9) reads

$$\Xi_{\mathrm{loop}}=\sum_{N=0}^{\infty}\frac{1}{N!}\int\prod_{n=1}^{N}[z(\mathscr{L}_n)d\mathscr{L}_n]\times\exp\left(-\beta\frac{1}{2}\sum_{I\neq J}e_{\alpha_I}e_{\alpha_J}v(\mathscr{L}_I,\mathscr{L}_J)\right).\tag{3.11}$$

In (3.11) we use the convention that, if $N=0$, there is no $\mathscr{L}_n$ in the corresponding term of Ξ_{loop} and this term is merely equal to 1.

The loop $\mathscr{L}_I$ can be described by a closed curve $\boldsymbol{\Omega}_I$, which is parametrized by an abscissa τ ranging from 0 to p_I,

$$\boldsymbol{\Omega}_I(\tau)=\sum_{l=1}^{p_I}\delta_{P(\tau),l-1}\boldsymbol{\omega}_l^I[\tau-(l-1)].\tag{3.12}$$

In (3.12) $P(\tau)$ denotes the integral part of τ [for instance, $\boldsymbol{\Omega}_I(\tau=0)\equiv\boldsymbol{\omega}_1^I(s=0)=\mathbf{x}_1^I$] and we set $\boldsymbol{\Omega}_I(\tau=p_I)=\lim_{\tau\to p_I^-}\omega_{p_I}^I[\tau-(p_I-1)]=\boldsymbol{\omega}_{p_I}^I(s=1)=\mathbf{x}_1^I$. The potential between loops (3.8) can be expressed as

$$v(\mathscr{L}_I,\mathscr{L}_J)=\int_0^{p_I}d\tau\int_0^{p_J}d\tau'\,\delta([\tau-P(\tau)]-[\tau'-P(\tau')])v_C[\boldsymbol{\Omega}_I(\tau)-\boldsymbol{\Omega}_J(\tau')].\tag{3.13}$$

It is represented in Fig. 1. The internal energy E_β^{int} (3.7) is the sum of two terms: the internal energy of a noninteracting loop $E_\beta^0(\mathscr{L}_I)$ and a self-interaction energy $E_{\mathrm{self}}(\mathscr{L}_I)$. The former reads

$$E_\beta^0(\mathscr{L}_I)=E_\beta^0(\{\mathbf{x}_l^I\})\equiv\frac{1}{\beta 2\lambda_{\alpha_I}^2}\sum_{l=1}^{p_I}(\mathbf{x}_{l+1}^I-\mathbf{x}_l^I)^2.\tag{3.14}$$

For noninteracting loops, the average extent of the curve $\boldsymbol{\Omega}_I$, i.e., the average distances $|\mathbf{x}_{l+1}^I-\mathbf{x}_l^I|$ and the average extent of the Brownian bridges $\lambda_{\alpha_I}\boldsymbol{\xi}_l^I$, with $l=1,\dots,p_I$, increases when the temperature is lowered. The self-interaction energy can be written as

$$E_{\mathrm{self}}(\mathscr{L}_I)=\tfrac{1}{2}e_\alpha^2\int_0^{p_I}d\tau\int_0^{p_I}d\tau'(1-\delta_{P(\tau),P(\tau')})\,\delta([\tau-P(\tau)]-[\tau'-P(\tau')])v_C[\boldsymbol{\Omega}_I(\tau)-\boldsymbol{\Omega}_I(\tau')].\tag{3.15}$$

The self-interaction energy $E_{\mathrm{self}}(\mathscr{L}_I)$ is different from the self-energy of a classical loop, which is equal to $\frac{1}{2}e_\alpha^2\int_0^{p_I}d\tau\int_0^{p_I}d\tau'[1-\delta(\tau-\tau')]v_C[\boldsymbol{\Omega}_I(\tau)-\boldsymbol{\Omega}_I(\tau')]$, but, as the latter, it does not contain any short-distance singularity. Since all the particles inside a loop have the same charge and since the potential v_C is a positive function of the distance, the self-interaction energy is positive and the internal energy of a loop (3.7) is larger than in the noninteracting case, whatever the signs of the particles are. Henceforth, according to (3.5), the absolute value of the fugacity of the interacting loops is lower than that of the noninteracting loops with the same chemical potential μ_α,

$$|z(\mathscr{L}_I)|<|z^0(\mathscr{L}_I)|=|z_{\alpha,p}^{*}|e^{-\beta E_\beta^0(\{\mathbf{x}_l^I\})}.\tag{3.16}$$

The weight of loops with many interacting exchanged particles is lowered with respect to its value in the noninteracting case. In some sense, the repulsive interaction between identical particles reduces the exchange effects.

The formula (3.11) can be checked readily in two cases: first, for a system of noninteracting particles, as shown in Sec IV B, and second, for any interacting system, in the semiclassical limit. Indeed, when $\hbar$ tends to zero without strictly vanishing, the internal energy (3.7) of every loop with a ‘‘size’’ $p\geqslant 2$ becomes infinite, because of the contribution $E^0_\beta(\mathscr{L}^{\alpha,p})$, and the weight $z(\mathscr{L}^{\alpha,p})$ becomes exponentially small. In such a regime, the exchange effects are negligible: there is no exchange loop with a size $p\geqslant 2$, but only closed Brownian ‘‘filaments’’ $\mathscr{L}^{\alpha,1}=(\alpha,\mathbf{r},\boldsymbol{\xi})$ with shapes $\boldsymbol{\xi}$. In the formula (3.11), the summation over p is reduced to its first term $p=1$ and $\int d\mathscr{L}\, z(\mathscr{L})$ is merely equal to $\Sigma_{\alpha=1}^{n_s}\int d\mathbf{r}\int D(\boldsymbol{\xi})z^*_{\alpha,1}$. Thus we retrieve the expression of Ref. [25] for the grand partition function in Maxwell-Boltzmann statistics. The latter was derived directly from the fact that the trace in the Maxwell-Boltzmann grand partition function is reduced to diagonal matrix elements, which correspond only to closed filaments interacting through the potential

$$v(\mathscr{L}^{\alpha_I,1},\mathscr{L}^{\alpha_J,1})=\int_0^1 ds\; v[\mathbf{r}_I-\mathbf{r}_J+\lambda_{\alpha_I}\boldsymbol{\xi}_I(s)-\lambda_{\alpha_J}\boldsymbol{\xi}_J(s)]. \tag{3.17}$$

In the strict classical limit $\hbar=0$, $\lambda_\alpha\boldsymbol{\xi}=\mathbf{0}$ and $v(\mathscr{L}^{\alpha_I,1},\mathscr{L}^{\alpha_J,1})=v(\mathbf{r}_I-\mathbf{r}_J)$; since $\int D(\boldsymbol{\xi})=1$, the Brownian bridge $\boldsymbol{\xi}$ disappears in the expression of Ξ and we retrieve the classical grand partition function.

For conciseness, we introduce the following new variables. We call $\mathbf{R}_I\equiv\mathbf{x}_1^I$ the ‘‘position’’ of the loop $\mathscr{L}_I$ [we could also have chosen the centroid $\mathbf{R}_I=(1/p)\Sigma_{l=1}^{p}\mathbf{x}_l^I$]. The ‘‘shape’’ of the loop is

$$\mathbf{X}_I(\tau)\equiv\boldsymbol{\Omega}_I(\tau)-\mathbf{R}_I=\sum_{l=1}^{p_I}\delta_{P(\tau),l-1}\left[\left(\sum_{l'=1}^{l-1}\mathbf{t}_{l'}^I\right)+[\tau-P(\tau)]\mathbf{t}_l^I\right.$$
$$\left.+\lambda_{\alpha_I}\boldsymbol{\xi}_l^I[\tau-P(\tau)]\right], \tag{3.18}$$

where the segments $\mathbf{t}_l^I$ join the positions of the particles in the loop, $\mathbf{t}_l^I\equiv\mathbf{x}_{l+1}^I-\mathbf{x}_l^I$ for $l=1,\ldots,p$ $(\mathbf{t}_p^I=\mathbf{x}_1^I-\mathbf{x}_p^I)$ and $\Sigma_{l=1}^{p}\mathbf{t}_l^I=\mathbf{0}$. The internal energy $E_\beta^{\mathrm{int}}(\mathscr{L}_I)$ of a loop $\mathscr{L}_I$ depends only on the internal degrees of freedom that include the species α_I, the size p_I, and the shape $\mathbf{X}_I$ of the loop. The integration measure $\int d\mathscr{L}$ can be decomposed into a contribution from the position of the loop and a contribution from the internal degrees of freedom $\int d\mathscr{L}=\int d\mathbf{R}\int D(\mathbf{X})$ with

$$D(\mathbf{X})\equiv\prod_{l=2}^{p}d\mathbf{x}_l\prod_{l=1}^{p}D(\boldsymbol{\xi}_l)=\prod_{l=1}^{p}d\mathbf{t}_l\,\delta\!\left(\sum_{l=1}^{p}\mathbf{t}_l\right)\prod_{l=1}^{p}D(\boldsymbol{\xi}_l). \tag{3.19}$$

(We mention that, in the Maxwell-Boltzmann approximation, where there are only loops with size $p=1$, each loop is reduced to a Brownian bridge with a shape $\boldsymbol{\xi}$ located at $\mathbf{r}$, the position of the sole corresponding quantum particle; then $\mathbf{R}$ is replaced by $\mathbf{r}$ and $\mathbf{X}$ by $\boldsymbol{\xi}$.)

C. Multipolar decomposition of the loop potential

In order to exhibit the difference between the loop interaction and the interaction between classical extended objects, we introduce a ‘‘multipolar’’ decomposition of the potential v defined in (3.13). We start from the Taylor expansion of v that exhibits the asymptotic behavior of v at large distances,

$$v(\mathscr{L}_i,\mathscr{L}_j)=p_ip_jv_C(R_{ij})+\int_0^{p_i}d\tau\int_0^{p_j}d\tau'$$
$$\times\,\delta([\tau-P(\tau)]-[\tau'-P(\tau')])\sum_{n=1}^{\infty}\frac{1}{n!}$$
$$\times[\mathbf{X}_i(\tau)\cdot\boldsymbol{\nabla}_i+\mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_j]^n v_C(R_{ij}), \tag{3.20}$$

where $\boldsymbol{\nabla}_i$ denotes $\boldsymbol{\nabla}_{\mathbf{R}_i}$. (In the following, a loop is indexed either by a capital I or by a small i.) At large distances, the loop potential $v(\mathscr{L}_i,\mathscr{L}_j)$ behaves as the Coulomb potential between the ‘‘total charges’’ of the two loops, as if they were point charges located at $\mathbf{R}_i$ and $\mathbf{R}_j$, respectively. In the following, the ‘‘total charge’’ of a loop $\mathscr{L}_i$ refers to the sum $p_ie_{\alpha_i}$ of the charges of the corresponding p_i quantum particles of species α_i and we use the argument c_i instead of $\mathscr{L}_i$ in the functions that involve only the position $\mathbf{R}_i$ and the total charge $p_ie_{\alpha_i}$ of the ith loop. For instance, the charge-charge potential between loops reads

$$v^{cc}(c_i,c_j)\equiv p_ip_jv_C(R_{ij}). \tag{3.21}$$

The multipolar decomposition (3.20) of the potential v can be reorganized as the sum of a charge-charge potential $v^{cc}(c_i,c_j)$, multipole-charge potentials $v^{mc}(\mathscr{L}_i,c_j)$ and $v^{cm}(c_i,\mathscr{L}_j)$ and a multipole-multipole potential $v^{mm}(\mathscr{L}_i,\mathscr{L}_j)$,

$$v(\mathscr{L}_i,\mathscr{L}_j)=v^{cc}(c_i,c_j)+v^{mc}(\mathscr{L}_i,c_j)+v^{cm}(c_i,\mathscr{L}_j)$$
$$+v^{mm}(\mathscr{L}_i,\mathscr{L}_j), \tag{3.22}$$

with the following definitions.

(a) $v^{mc}(\mathscr{L}_i,c_j)$ is the sum of the interactions between the total charge of the loop $\mathscr{L}_j$, as if it were concentrated at $\mathbf{R}_j$, and all the multipolar moments of order q (with $q\geqslant 1$) of a charged filament that would have the shape of the loop $\mathscr{L}_i$ and a charge density $\sigma_i(\mathbf{r})\equiv\int_0^{p_i}d\tau\;\delta(\mathbf{r}-\boldsymbol{\Omega}_i(\tau))$,

$$v^{mc}(\mathscr{L}_i,c_j)\equiv p_j\int_0^{p_i}d\tau\sum_{q=1}^{\infty}\frac{1}{q!}[\mathbf{X}_i(\tau)\cdot\boldsymbol{\nabla}_i]^q v_C(R_{ij})$$
$$=v_{\mathrm{elect}}(\mathscr{L}_i,c_j)-v^{cc}(c_i,c_j), \tag{3.23}$$

where $v_{\mathrm{elect}}(\mathscr{L}_i,c_j)$ is the classical electrostatic potential between a point charge c_j and the charged closed line $\boldsymbol{\Omega}_i(\tau)=\mathbf{R}_i+\mathbf{X}_i(\tau)$,

$$v_{\mathrm{elect}}(\mathscr{L}_i,c_j)\equiv p_j\int_0^{p_i}d\tau\; v_C[\boldsymbol{\Omega}_i(\tau)-\mathbf{R}_j]$$
$$=p_j\int d\mathbf{r}\;\sigma_i(\mathbf{r})v_C(\mathbf{r}-\mathbf{R}_j). \tag{3.24}$$

(b) The second term is similar to the first one, with the indices i and j interchanged,

$$v^{cm}(c_i,\mathscr{L}_j)\equiv p_i\int_0^{p_j}d\tau'\sum_{q=1}^{\infty}\frac{1}{q!}\,[\mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_j]^q v_C(R_{ij})$$
$$=v_{\text{elect}}(c_i,\mathscr{L}_j)-v^{cc}(c_i,c_j). \qquad (3.25)$$

(c) The last contribution $v^{mm}(\mathscr{L}_i,\mathscr{L}_j)$ is the sum of the interactions between the multipolar moments of order q ($q\geqslant 1$) of every infinitesimal line element of the curve $\boldsymbol{\Omega}_i$ with parameter τ and the multipolar moments of order q' ($q'\geqslant 1$) of the p infinitesimal line elements of $\boldsymbol{\Omega}_j$ with parameter τ' such that $\tau-P(\tau)=\tau'-P(\tau')$,

$$v^{mm}(\mathscr{L}_i,\mathscr{L}_j)=\int_0^{p_i}d\tau\int_0^{p_j}d\tau'\,\delta([\tau-P(\tau)]$$
$$-[\tau'-P(\tau')])\sum_{q=1}^{\infty}\sum_{q'=1}^{\infty}\frac{1}{q!}\frac{1}{q'!}$$
$$\times[\mathbf{X}_i(\tau)\cdot\boldsymbol{\nabla}_i]^q[\mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_j]^{q'}v_C(R_{ij}). \qquad (3.26)$$

This part of the loop interaction has no classical interpretation because, in electrostatics, any infinitesimal part of a charged line interacts with every infinitesimal line element of another charged curve. Indeed, the electrostatic potential between classical loops reads

$$v_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j)=\int_0^{p_i}d\tau\int_0^{p_j}d\tau'\,v_C[\boldsymbol{\Omega}_i(\tau)-\boldsymbol{\Omega}_j(\tau')]$$
$$=\int d\mathbf{r}\int d\mathbf{r}'\,\sigma_i(\mathbf{r})\sigma_j(\mathbf{r}')v_C(\mathbf{r}-\mathbf{r}'). \qquad (3.27)$$

From the Taylor expansion of (3.27), we derive a multipolar decomposition of v_{elect} analogous to (3.22),

$$v_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j)=v^{cc}(c_i,c_j)+v^{mc}(\mathscr{L}_i,c_j)+v^{cm}(c_i,\mathscr{L}_j)$$
$$+v^{mm}_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j), \qquad (3.28)$$

with

$$v^{mm}_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j)=\int_0^{p_i}d\tau\int_0^{p_j}d\tau'\sum_{q=1}^{\infty}\sum_{q'=1}^{\infty}\frac{1}{q!}\frac{1}{q'!}$$
$$\times[\mathbf{X}_i(\tau)\cdot\boldsymbol{\nabla}_i]^q[\mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_j]^{q'}v_C(R_{ij}). \qquad (3.29)$$

The difference between the potentials v and v_{elect} is just $v(\mathscr{L}_i,\mathscr{L}_j)-v_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j)=v^{mm}(\mathscr{L}_i,\mathscr{L}_j)-v^{mm}_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j)$. As a final remark, we stress that the above decompositions are independent from the particular form of the two-body potential between the quantum point particles.

D. Loop distributions

The density $\rho(\mathscr{L})\equiv\rho(\mathscr{L}^{\alpha,p})$ of loops of species α, size p, "located" at $\mathbf{x}_1$ and with a shape $(\{\mathbf{x}_l-\mathbf{x}_1\}_{l=2,\ldots,p}$, $\{\boldsymbol{\xi}_l\}_{l=1,\ldots,p})$ is the average of the operator

$$\hat{\rho}(\mathscr{L})\equiv\hat{\rho}(\alpha,p,\{\mathbf{x}_l\}_{l=1,\ldots,p},\ \{\xi_l\}_{l=1,\ldots,p})$$
$$=\sum_I\delta_{\alpha_I,\alpha}\delta_{p_I,p}\prod_{l=1}^{p}\delta(\mathbf{x}_l^I-\mathbf{x}_l)\prod_{l=1}^{p}\delta(\boldsymbol{\xi}_l^I-\boldsymbol{\xi}_l). \qquad (3.30)$$

We notice that, in the case of fermions, since the sign of the weight $z(\mathscr{L})$ depends on the size p of the loop according to (3.6), $\rho(\mathscr{L})$ is expected not to be positive for every p, as is the case for noninteracting fermions (see Sec. IV B). The correlation $\rho^{(2)T}(\mathscr{L}_a,\mathscr{L}_b)$ between the loops $\mathscr{L}_a$ and $\mathscr{L}_b$, with $\mathscr{L}_a=(\alpha_a,p_a,\{\mathbf{x}_l^a\}_{l=1,\ldots,p_a},\ \{\boldsymbol{\xi}_l^a\}_{l=1,\ldots,p_a})$, is the truncated average of the two-body distribution operator

$$\rho^{(2)T}(\mathscr{L}_a,\mathscr{L}_b)+\rho(\mathscr{L}_a)\rho(\mathscr{L}_b)$$
$$=\left\langle\sum_{I\neq J}\delta_{\alpha_I,\alpha_a}\delta_{\alpha_J,\alpha_b}\delta_{p_I,p_a}\delta_{p_J,p_b}\prod_{l=1}^{p_a}\delta(\mathbf{x}_l^I-\mathbf{x}_l^a)\right.$$
$$\left.\times\prod_{l'=1}^{p_b}\delta(\mathbf{x}_{l'}^J-\mathbf{x}_{l'}^b)\prod_{l=1}^{p_a}\delta(\boldsymbol{\xi}_l^I-\boldsymbol{\xi}_l^a)\prod_{l'=1}^{p_b}\delta(\boldsymbol{\xi}_{l'}^J-\boldsymbol{\xi}_{l'}^b)\right\rangle. \qquad (3.31)$$

According to (3.10) and (3.11), Ξ_{loop} can be seen as a functional of $z(\mathscr{L})$, where each loop $\mathscr{L}_I$ has a weight $z(\mathscr{L}_I)$. Thus the distribution functions of the loops can be derived as functional derivatives of the grand partition function Ξ_{loop} with respect to $z(\mathscr{L})$,

$$\rho(\mathscr{L})=z(\mathscr{L})\,\frac{\delta(\ln\Xi_{\text{loop}})}{\delta(z(\mathscr{L}))} \qquad (3.32)$$

and, for $\mathscr{L}_a\neq\mathscr{L}_b$,

$$\rho^{(2)T}(\mathscr{L}_\alpha,\mathscr{L}_b)=z(\mathscr{L}_a)z(\mathscr{L}_b)\,\frac{\delta^2(\ln\Xi)}{\delta(z(\mathscr{L}_a))\delta(z(\mathscr{L}_b))}. \qquad (3.33)$$

Because of the noncommutativity of the operators $\hat{\rho}(\mathbf{r})$ and $\hat{H}$, such functional relations do not exist in usual other formalisms describing a quantum system. They allow one to build Mayer-like diagrams, as shown in Sec. VI.

IV. QUANTUM PARTICLE VERSUS EXCHANGE LOOP DISTRIBUTIONS

A. Density and correlations

The density operator $\hat{\rho}_\alpha(\mathbf{r})$ (2.5) for the quantum particles of species α can be expressed as either a sum over the particle indices i or a double sum over the loop indices I and the indices l of the particle variables $\mathbf{x}_l^I$. The analog of (3.30) reads

$$\hat{\rho}_\alpha(\mathbf{r})=\sum_I\delta_{\alpha_I,\alpha}\sum_{l=1}^{p_I}\delta(\mathbf{x}_l^I-\mathbf{r}). \qquad (4.1)$$

A comparison of (3.30) and (4.1) gives the relation between the particle- and loop-density operators

$$\hat{\rho}_\alpha(\mathbf{r})=\sum_{p=1}^{\infty}\int\prod_{l=1}^{p}d\mathbf{x}_l\sum_{l=1}^{p}\delta(\mathbf{x}_l-\mathbf{r})\int\prod_{l=1}^{p}D(\boldsymbol{\xi}_l)\hat{\rho}(\mathscr{L}^{\alpha,p})$$
$$=\sum_{p=1}^{\infty}p\int\prod_{l=2}^{p}d\mathbf{x}_l\int\prod_{l=1}^{p}D(\boldsymbol{\xi}_l)\hat{\rho}(\mathscr{L}^{\alpha,p})|_{\mathbf{x}_1=\mathbf{r}}. \quad (4.2)$$

The last equality in (4.2) is derived from the invariance of $\hat{\rho}(\mathscr{L}^{\alpha,p})$ under any cyclic permutation of the p positions $\mathbf{x}_l$. In the infinite volume, the average density of loops $\rho(\mathscr{L}^{\alpha,p})=\langle\hat{\rho}(\mathscr{L}^{\alpha,p})\rangle$ depends only on $p-1$ independent positions $\mathbf{x}_l$ and p Brownian bridges $\boldsymbol{\xi}_l$, $\rho(\mathscr{L}^{\alpha,p})=\rho_{\alpha,p}(\mathbf{X})$. Thus the density of quantum particles can be deduced from the density of loops by

$$\rho_\alpha^Q(\mathbf{r})=\sum_{p=1}^{\infty}p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X}), \quad (4.3)$$

with the notation of (3.19).

In the same way, the two-body distribution operator is written in terms of the loop indices I as

$$\sum_{i\neq j}\delta_{\alpha_i,\alpha_a}\delta_{\alpha_j,\alpha_b}\delta(\mathbf{r}_i-\mathbf{r}_a)\delta(\mathbf{r}_j-\mathbf{r}_b)$$
$$=\delta_{\alpha_a,\alpha_b}\sum_{I}\delta_{\alpha_I,\alpha_a}\sum_{l\neq l*}\delta(\mathbf{x}_l^I-\mathbf{r}_a)\delta(\mathbf{x}_{l*}^I-\mathbf{r}_b)$$
$$+\sum_{I\neq J}\delta_{\alpha_I,\alpha_a}\delta_{\alpha_J,\alpha_b}\sum_{l=1}^{p_I}\delta(\mathbf{x}_l^I-\mathbf{r}_a)\sum_{l*=1}^{p_J}\delta(\mathbf{x}_{l*}^J-\mathbf{r}_b). \quad (4.4)$$

The first term on the right-hand side of (4.4) comes from particles that belong to the same loop, whereas the second term is the contribution from particles that are in two different loops. As a consequence, the correlation function of two particles (2.6) can be written as the sum of the contributions from configurations where the particles at the two considered positions are either exchanged or not exchanged within the same cycle,

$$\rho_{\alpha_a\alpha_b}^{(2)TQ}(\mathbf{r}_{ab})=\delta_{\alpha_a,\alpha_b}\rho_{\alpha_a\alpha_a}^{(2)TQ}|_{\text{exch}}(\mathbf{r}_{ab})+\rho_{\alpha_a\alpha_b}^{(2)TQ}|_{\text{nonexch}}(\mathbf{r}_{ab}), \quad (4.5)$$

with

$$\rho_{\alpha\alpha}^{(2)TQ}|_{\text{exch}}(\mathbf{r}_{ab})=\sum_{p=2}^{\infty}p\int\prod_{l=1}^{p}d\mathbf{x}_l\int\prod_{l=1}^{p}D(\boldsymbol{\xi}_l)\rho(\mathscr{L}^{\alpha,p})$$
$$\times\delta(\mathbf{x}_1-\mathbf{r}_a)\sum_{l*=2}^{p}\delta(\mathbf{x}_{l*}-\mathbf{x}_1-\mathbf{r}_{ab})$$
$$=\sum_{p=2}^{\infty}p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})$$
$$\times\sum_{l*=2}^{p}\delta\left(\mathbf{r}_{ab}-\sum_{l=1}^{l*-1}\mathbf{t}_l\right) \quad (4.6)$$

and

$$\rho_{\alpha_a\alpha_b}^{(2)TQ}|_{\text{nonexch}}(\mathbf{r}_{ab})=\sum_{p=1}^{\infty}\sum_{p'=1}^{\infty}pp'\int\prod_{l=2}^{p}d\mathbf{x}_l\int\prod_{l=1}^{p}D(\boldsymbol{\xi}_l)$$
$$\times\int\prod_{l'=2}^{p'}d\mathbf{x}'_{l'}\int\prod_{l'=1}^{p'}D(\boldsymbol{\xi}'_{l'})$$
$$\times\rho^{(2)T}(\mathscr{L}^{\alpha_a,p},\mathscr{L}'^{\alpha_b,p'})|_{\mathbf{x}_1=\mathbf{r}_a,\mathbf{x}'_1=\mathbf{r}_b}$$
$$=\sum_{p_a=1}^{\infty}\sum_{p_b=1}^{\infty}p_ap_b\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)$$
$$\times\rho_{\alpha_a,p_a;\alpha_b,p_b}^{(2)T}(\mathbf{r}_{ab};\mathbf{X}_a,\mathbf{X}_b). \quad (4.7)$$

Indeed, $\rho^{(2)T}(\mathscr{L}^{\alpha_a,p},\mathscr{L}'^{\alpha_b,p'})$ is invariant under any cyclic permutation of the p positions $\mathbf{x}_l$ or the p' positions $\mathbf{x}'_{l'}$. Moreover, in the thermodynamic limit, the bulk correlation between two loops $\mathscr{L}_a$ and $\mathscr{L}_b$ depends only on p_a+p_b-1 independent positions and p_a+p_b Brownian bridges, $\rho^{(2)T}(\mathscr{L}_a,\mathscr{L}_b)=\rho_{\alpha_a,p_a;\alpha_b,p_b}^{(2)T}(\mathbf{r}_{ab};\mathbf{X}_a,\mathbf{X}_b)$.

Equations (4.5) and (4.6) imply that

$$\int d\mathbf{r}\,\rho_{\alpha\alpha}^{(2)TQ}(\mathbf{r})=\sum_{p=1}^{\infty}p(p-1)\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})$$
$$+\int d\mathbf{r}\,\rho_{\alpha\alpha}^{(2)TQ}|_{\text{nonexch}}(\mathbf{r}). \quad (4.8)$$

By using (4.3), (4.5), and the Orsntein-Zernicke relation (2.10), we get the identities

$$\sum_{p=1}^{\infty}p^2\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})=\rho_\alpha^Q+\int d\mathbf{r}\,\rho_{\alpha\alpha}^{(2)TQ}|_{\text{exch}}(\mathbf{r})$$
$$=\left.\frac{\partial\rho_\alpha^Q}{\partial(\beta\mu_\alpha)}\right|_\beta$$
$$-\int d\mathbf{r}\,\rho_{\alpha\alpha}^{(2)TQ}|_{\text{nonexch}}(r_{ab}). \quad (4.9)$$

B. Noninteracting system

For an ideal gas, the contributions from the noninteracting loops are factored in the grand partition function (3.11)

$$\Xi_0(\beta,\{\mu_\alpha\},\Lambda)=\exp\left(\int_\Lambda d\mathscr{L}\,z_0(\mathscr{L})\right). \quad (4.10)$$

Moreover, the internal energy (3.7) of free loops does not depend on the Brownian paths, so that

$$\int_\Lambda d\mathscr{L}\,z_0(\mathscr{L})=\sum_{\alpha}^{n_s}\sum_{p=1}^{\infty}z_{\alpha,p}^*\int_\Lambda\prod_{l=1}^{p}d\mathbf{x}_l e^{-\beta E_\beta^0(\{\mathbf{x}_l\})}\prod_{l=1}^{p}$$
$$\times\int D(\boldsymbol{\xi}_l). \quad (4.11)$$

The measure $D(\boldsymbol{\xi})$ is normalized to $\int D(\boldsymbol{\xi})=1$. In order to perform the integration over the variables $\mathbf{x}_l$, we use the change of variables (3.18) $\mathbf{t}_l = \mathbf{x}_{l+1} - \mathbf{x}_l$ (with the convention $\mathbf{x}_{p+1} \equiv \mathbf{x}_1$) and the Fourier representation of the Dirac distribution

$$\lim_{\Lambda\to\infty} \frac{1}{\Lambda} \int_\Lambda \prod_{l=1}^{p} d\mathbf{x}_l e^{-\beta E_\beta^0(\{\mathbf{x}_l\})}$$
$$= \int \prod_{l=1}^{p} d\mathbf{t}_l \delta\left(\sum_{l=1}^{p} \mathbf{t}_l \right) \prod_{l=1}^{p} e^{-(1/2\lambda_\alpha^2)\mathbf{t}_l^2}$$
$$= \int \frac{d\mathbf{k}}{(2\pi)^3} [(2\pi\lambda_\alpha^2)^{3/2} e^{-\lambda_\alpha^2 \mathbf{k}^2/2}]^p. \quad (4.12)$$

$\lambda_\alpha^2 \mathbf{k}^2/2$ is equal to β times the kinetic energy $\epsilon_\alpha^0(\mathbf{k})$ of a free particle with momentum $\mathbf{k}$. According to the definition of $z_{\alpha,p}^*$ (3.6), $\lim_{\Lambda\to\infty}[\ln(\Xi_0)/\Lambda]$ involves the series $\Sigma_{p=1}^{\infty}(1/p)[\eta_\alpha e^{\beta[\mu_\alpha - \epsilon_\alpha^0(\mathbf{k})]}]^p$. This series is convergent for the wave vectors $\mathbf{k}$ such that $\epsilon_\alpha^0(\mathbf{k}) > \mu_\alpha$ and the value of the sum is given by analytic continuation for all $\mathbf{k}$'s. Thus we retrieve the usual formula for the pressure (2.9) P_0 of free particles in an infinite volume

$$\beta P_0 = \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \ln \Xi_0 = \sum_{\alpha=1}^{n_s} (2S_\alpha + 1) \int \frac{d\mathbf{k}}{(2\pi)^3} (-\eta_\alpha)$$
$$\times \ln(1 - \eta_\alpha e^{\beta[\mu_\alpha - \epsilon_\alpha^0(\mathbf{k})]}). \quad (4.13)$$

The density of noninteracting loops, obtained from (3.32) and (4.10), is equal to the loop fugacity, as in a classical ideal gas of point particles,

$$\rho^0(\mathscr{L}) = z^0(\mathscr{L}). \quad (4.14)$$

For fermions, the sign of $\rho^0(\mathscr{L})$ depends on p, as is the case for the sign of $z_{\alpha,p}^*$ (3.6). The density of free quantum particles is given by (4.2)

$$\rho_\alpha^{0Q} = \sum_{p=1}^{\infty} p z_{\alpha,p}^* \int \prod_{l=2}^{p} d\mathbf{x}_l e^{-\beta E_\beta^0(\{\mathbf{x}_l\})} \int \prod_{l=1}^{p} D(\boldsymbol{\xi}_l). \quad (4.15)$$

Since only $z_{\alpha,p}^*$ depends on μ_α in (4.15) and $\partial z_{\alpha,p}^*/\partial(\beta\mu_\alpha) = p z_{\alpha,p}^*$, it can be checked, by comparing (4.11) and (4.15), that ρ_α^{0Q} is directly deduced from the expression (4.13) of the pression P_0 by the usual thermodynamic relation (2.8)

$$\rho_\alpha^{0Q} = \frac{\partial(\beta P^0)}{\partial(\beta\mu_\alpha)} = (2S_\alpha + 1) \int \frac{d\mathbf{k}}{(2\pi)^3} n_\alpha^0(\mathbf{k}). \quad (4.16)$$

$n_\alpha^0(\mathbf{k})$ is the occupation number of the state with quantum numbers α and $\mathbf{k}$,

$$n_\alpha^0(\mathbf{k}) = \frac{e^{\beta[\mu_\alpha - \epsilon_\alpha^0(\mathbf{k})]}}{1 - \eta_\alpha e^{\beta[\mu_\alpha - \epsilon_\alpha^0(\mathbf{k})]}} = \frac{1}{e^{\beta[\epsilon_\alpha^0(\mathbf{k}) - \mu_\alpha]} - \eta_\alpha}. \quad (4.17)$$

We recall that, in fact, the integral in (4.16) is to be understood as the limit of a sum over the discrete quantum wave numbers. Indeed, in the case of bosons, when the temperature is lower than the Bose-condensation critical value, $\mu_\alpha = 0$ and the density must be written as

$$\rho_\alpha^{0Q} = \rho_{\alpha,G}^{0Q} + (2S_\alpha + 1) \lim_{\epsilon\to 0} \int_{\epsilon < |\mathbf{k}|} \frac{d\mathbf{k}}{(2\pi)^3} n_\alpha^0(\mathbf{k}; \beta, \mu_\alpha = 0),$$

where $\rho_{\alpha,G}^{0Q}$ is the contribution from the ground state [35].

The correlation function between the noninteracting loops vanishes

$$\rho^{(2)T0}(\mathscr{L}, \mathscr{L}') = 0, \quad (4.18)$$

as can be checked by using (3.33) and (4.10). According to (4.5), the correlation $\rho_{\alpha\alpha}^{(2)T0Q}(r_{ab})$ between free quantum particles is reduced to the contribution from the density of noninteracting loops of the same species. In other words, the correlation in an ideal quantum gas contains only the purely statistical exchange term

$$\rho_{\alpha\alpha'}^{(2)T0Q}(r_{ab}) = \delta_{\alpha,\alpha'} \rho_{\alpha\alpha}^{(2)T0Q}|_{\text{exch}}(r_{ab}) \quad (4.19)$$

and, according to (4.6) and (4.14),

$$\rho_{\alpha\alpha}^{(2)T0Q}(r_{ab}) = \eta_\alpha (2S_\alpha + 1) \sum_{p=2}^{\infty} \left(\frac{\eta_\alpha e^{\beta\mu_\alpha}}{(2\pi\lambda_\alpha^2)^{3/2}} \right)^p$$
$$\times \sum_{l^*=2}^{p} \int \prod_{l=1}^{p} d\mathbf{t}_l \delta\left(\mathbf{r}_{ab} - \sum_{l=1}^{l^*-1} \mathbf{t}_l \right)$$
$$\times \delta\left(\mathbf{r}_{ab} + \sum_{l=l^*}^{p} \mathbf{t}_l \right) \exp\left(-\frac{1}{2\lambda_\alpha^2} \sum_{l=1}^{p} \mathbf{t}_l^2 \right).$$
$$(4.20)$$

The Fourier transform representations of the Dirac distributions are used to perform the integration over the variables $\mathbf{t}_l$, with the result

$$\rho_{\alpha\alpha}^{(2)T0Q}(r_{ab}) = \eta_\alpha (2S_\alpha + 1) \int_\infty \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{r}_{ab}} \int \frac{d\mathbf{k}'}{(2\pi)^3}$$
$$\times e^{-i\mathbf{k}'\cdot\mathbf{r}_{ab}} \sum_{p=2}^{\infty} \sum_{l^*=2}^{p} [\eta_\alpha e^{[\beta\mu_\alpha - (\lambda_\alpha^2/2)\mathbf{k}^2]}]^{l^*-1}$$
$$\times [\eta_\alpha e^{[\beta\mu_\alpha - (\lambda_\alpha^2/2)\mathbf{k}'^2]}]^{p-l^*+1}$$
$$= \eta_\alpha (2S_\alpha + 1) \left(\int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{r}_{ab}} \right.$$
$$\left. \times \sum_{q=1}^{\infty} [\eta_\alpha e^{\beta[\mu_\alpha - \epsilon_\alpha^0(\mathbf{k})]}]^q \right)^2. \quad (4.21)$$

The last equality in (4.21) is obtained by exchanging the order of the summations over p and l^* and changing $p-l^*+1$ into q and l^*-1 into q'. The sum of the series over q is merely equal to η_α times the occupation number $n_\alpha^0(\mathbf{k})$ (4.17). Eventually

$$\rho_{\alpha\alpha}^{(2)T0Q}(r_{ab})=\eta_\alpha(2S_\alpha+1)\left(\int\frac{d\mathbf{k}}{(2\pi)^3}\,e^{-i\mathbf{k}\cdot\mathbf{r}_{ab}}n_\alpha^0(\mathbf{k})\right)^2. \tag{4.22}$$

The integral in (4.22) is the well-known off-diagonal matrix element of the one-body density matrix for a noninteracting gas. It is equal to $\lim_{s\to 0-}G_\alpha^0(\mathbf{r}_{ab},s)=\int[d\mathbf{k}/(2\pi)^3]\exp[-i\mathbf{k}\cdot\mathbf{r}_{ab}]n_\alpha^0(\mathbf{k})$, where $G_\alpha^0(\mathbf{r}_{ab},s)$ is the free single-particle Green's function of the standard perturbation many-body formalism [see (6.1) in Sec. VI]. As already mentioned previously about Eq. (4.16), in the Bose-condensation phase, the integral over $\mathbf{k}$ must be written as $\rho_{\alpha,G}^{0Q}+\lim_{\epsilon\to 0}\int_{\epsilon<|\mathbf{k}|}\cdots$.

Since the noninteracting loops are not correlated, according to (4.18), the corresponding Ornstein-Zernicke relation (4.9) in terms of the free loop density reads

$$\sum_{p=1}^{\infty}p^2\int D(\mathbf{X})\rho_{\alpha,p}^0(\mathbf{X})=\frac{\partial\rho_\alpha^{0Q}}{\partial(\beta\mu_\alpha)}=\frac{[\rho_\alpha^{0Q}]^2\chi_{\alpha,T}^0}{\beta}, \tag{4.23}$$

where $\chi_{\alpha,T}^0$ is the isothermal compressibility of the quantum ideal gas.

C. Linear response to an external charge

When an infinitesimal distribution of charge $\delta q(\mathbf{r})$ is immersed into the system, the Hamiltonian $\hat{H}_{\{N_\alpha\}}$ in the grand partition function (2.3) is changed into $\hat{H}_{\{N_\alpha\}}+\int d\mathbf{r}'\,\delta q(\mathbf{r}')\Sigma_i e_{\alpha_i}v_C(\hat{\mathbf{r}}_i-\mathbf{r}')$, where $\hat{\mathbf{r}}_i$ is the position operator of the ith particle. The integrand involving the Brownian paths $\boldsymbol{\omega}_{i,\pi(i)}$ in the Feynman-Kac formula (3.2) is multiplied by

$$\exp\left(-\beta\int d\mathbf{r}'\,\delta q(\mathbf{r}')\sum_i e_{\alpha_i}\int_0^1 ds\; v_C(\boldsymbol{\omega}_{i,\pi(i)}-\mathbf{r}')\right).$$

Subsequently, Ξ_{loop} is changed into $\Xi_{\text{loop}}^{\text{ext}}$, where the Boltzmann factor of (3.11) is multiplied by $\exp[-\beta\hat{h}^{ext}]$ with

$$\hat{h}_{\text{ext}}=\int d\mathbf{r}'\,\delta q(\mathbf{r}')\int d\mathscr{L}'e_{\alpha'}\hat{\rho}(\mathscr{L}')\times\int_0^{p'}d\tau'v[\boldsymbol{\Omega}'(\tau')-\mathbf{r}'], \tag{4.24}$$

where $\hat{\rho}(\mathscr{L}')$ is given by (3.30). The induced density of loops $\rho^{\text{ind}}(\mathscr{L})\equiv\langle\hat{\rho}(\mathscr{L})\rangle_{\Xi_{\text{loop}}^{\text{ext}}}-\langle\hat{\rho}(\mathscr{L})\rangle_{\Xi_{\text{loop}}}$ can be derived by the usual linear response theory for classical systems

$$\begin{aligned}\rho^{\text{ind}}(\mathscr{L})&=-\beta[\langle\hat{\rho}(\mathscr{L})\hat{h}_{\text{ext}}\rangle-\langle\hat{\rho}(\mathscr{L})\rangle\langle\hat{h}_{\text{ext}}\rangle]\\&=-\beta\int d\mathbf{r}'\,\delta q(\mathbf{r}')\int d\mathscr{L}'e_{\alpha'}[\langle\hat{\rho}(\mathscr{L})\hat{\rho}(\mathscr{L}')\rangle\\&\quad-\langle\hat{\rho}(\mathscr{L})\rangle\langle\hat{\rho}(\mathscr{L}')\rangle]\int_0^{p'}d\tau'v[\boldsymbol{\Omega}'(\tau')-\mathbf{r}'].\end{aligned} \tag{4.25}$$

According to the definitions (3.30) and (3.31), $\langle\hat{\rho}(\mathscr{L})\hat{\rho}(\mathscr{L}')\rangle-\langle\hat{\rho}(\mathscr{L})\rangle\,\langle\hat{\rho}(\mathscr{L}')\rangle=\delta_{\mathscr{L},\mathscr{L}'}\rho(\mathscr{L})+\rho^{(2)T}(\mathscr{L},\mathscr{L}')$. By using the relation (4.2) between the particle-density and the loop-density operators, the induced charge density in the quantum system is found to read

$$\begin{aligned}\sum_\alpha e_\alpha\rho_\alpha^{\text{ind}Q}(\mathbf{r})=-\beta\int d\mathbf{r}'\,\delta q(\mathbf{r}')\Bigg\{&\sum_\alpha e_\alpha^2\sum_{p=1}^{\infty}p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})\int_0^p d\tau\;v[\mathbf{r}+\mathbf{X}(\tau)-\mathbf{r}']\\&+\sum_\alpha e_\alpha\sum_{\alpha'}e_{\alpha'}\int d\mathbf{R}'\sum_{p=1}^{\infty}p\sum_{p'=1}^{\infty}\int D(\mathbf{X})\int D(\mathbf{X}')\rho_{\alpha,p;\alpha',p'}^{(2)T}(\mathbf{r}-\mathbf{R}',\mathbf{X},\mathbf{X}')\\&\times\int_0^{p'}d\tau'v[\mathbf{R}'+\mathbf{X}'(\tau')-\mathbf{r}']\Bigg\}.\end{aligned} \tag{4.26}$$

In order to handle concise formulas, we will distinguish a function $g(\mathbf{r})$ from its Fourier transform only by their arguments, $g(\mathbf{k})\equiv\int d\mathbf{r}\exp(i\mathbf{k}\cdot\mathbf{r})g(\mathbf{r})$. The Fourier transform of the induced charge density is

$$\begin{aligned}\sum_\alpha e_\alpha\rho_\alpha^{\text{ind}Q}(\mathbf{k})=-\beta\delta q(\mathbf{k})v_C(\mathbf{k})\Bigg\{&\sum_\alpha e_\alpha^2\sum_{p=1}^{\infty}p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})\int_0^p d\tau e^{-i\mathbf{k}\cdot\mathbf{X}(\tau)}\\&+\sum_\alpha e_\alpha\sum_{\alpha'}e_{\alpha'}\sum_{p=1}^{\infty}p\sum_{p'=1}^{\infty}\int D(\mathbf{X})\int D(\mathbf{X}')\rho_{\alpha,p;\alpha',p'}^{(2)T}(\mathbf{k},\mathbf{X},\mathbf{X}')\int_0^{p'}d\tau'e^{-i\mathbf{k}\cdot\mathbf{X}'(\tau')}\Bigg\}.\end{aligned} \tag{4.27}$$

Formula (4.27) can be reexpressed in order to make the charge-charge correlation function appear. Since, according to (3.12), $\mathbf{X}(P(\tau))=\mathbf{x}_{P(\tau)+1}-\mathbf{x}_1$ and $\int_0^1 d\tau \exp[i\mathbf{k}\cdot\mathbf{X}(P(\tau))]=1$, we write

$$\sum_{l^*=2}^{p} e^{i\mathbf{k}\cdot(\mathbf{x}_{l^*}-\mathbf{x}_1)}=-1+\int_0^p d\tau\, e^{i\mathbf{k}\cdot\mathbf{X}(P(\tau))}. \tag{4.28}$$

Thus, according to (4.3), $\rho_{\alpha,p}(-\mathbf{X})=\rho_{\alpha,p}(\mathbf{X})$ and the Fourier transform of $\rho_{\alpha\alpha}^{(2)TQ}|_{\text{exch}}(\mathbf{r})$ (4.6) is found to read

$$\rho_{\alpha\alpha}^{(2)TQ}|_{\text{exch}}(\mathbf{k})=-\rho_\alpha^Q+\sum_{p=1}^{\infty} p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X}) \times\int_0^p d\tau\, e^{-i\mathbf{k}\cdot\mathbf{X}(P(\tau))}. \tag{4.29}$$

If $\mathbf{k}=\mathbf{0}$, we retrieve the identity (4.9). The charge-charge structure factor of the quantum particles of the medium (slightly improperly called charge-charge correlation in the following) is defined as

$$C^Q(\mathbf{r})\equiv\sum_{\alpha}\sum_{\alpha'} e_\alpha e_{\alpha'}[\delta_{\alpha,\alpha'}\rho_\alpha^Q\delta(\mathbf{r})+\rho_{\alpha\alpha'}^{(2)TQ}(\mathbf{r})] \tag{4.30}$$

and its Fourier transform is given by (4.5) and (4.29)

$$C^Q(\mathbf{k})=\sum_\alpha e_\alpha^2\sum_{p=1}^{\infty} p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})\int_0^p d\tau\, e^{-i\mathbf{k}\cdot\mathbf{X}(P(\tau))} +\sum_\alpha e_\alpha\sum_{\alpha'} e_{\alpha'}\sum_{p=1}^{\infty} p\sum_{p'=1}^{\infty} p'\int D(\mathbf{X})\int D(\mathbf{X}') \times\rho_{\alpha,p;\alpha',p'}^{(2)T}(\mathbf{k},\mathbf{X},\mathbf{X}'). \tag{4.31}$$

Therefore, the induced charge density (4.27) is related to the charge-charge correlation via

$$\frac{1}{\delta q(\mathbf{k})}\sum_\alpha e_\alpha\rho_\alpha^{\text{ind}Q}(\mathbf{k}) =-\beta v_C(\mathbf{k})\Bigg\{C^Q(\mathbf{k})+\sum_\alpha e_\alpha^2\sum_{p=1}^{\infty} p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X}) \times\int_0^p d\tau[e^{-i\mathbf{k}\cdot\mathbf{X}(\tau)}-e^{-i\mathbf{k}\cdot\mathbf{X}(P(\tau))}] +\sum_\alpha e_\alpha\sum_{\alpha'} e_{\alpha'}\sum_{p=1}^{\infty} p\sum_{p'=1}^{\infty}\int D(\mathbf{X}) \times\int D(\mathbf{X}')\rho_{\alpha,p;\alpha',p'}^{(2)T}(\mathbf{k},\mathbf{X},\mathbf{X}') \times\int_0^{p'} d\tau'[e^{-i\mathbf{k}\cdot\mathbf{X}'(\tau')}-1]\Bigg\}. \tag{4.32}$$

The small-$\mathbf{k}$ expansion of the induced charge density involves only even powers of $|\mathbf{k}|$ because of rotational invariance and the quantum correction term to $C^Q(\mathbf{k})$ inside the curly brackets in (4.32) starts at the order $|\mathbf{k}|^2$. In this expansion, the contributions from $p=1$ and $p'=1$ correspond to Maxwell-Boltzmann terms; since $\mathbf{X}(\tau)$ reduces to $\lambda_\alpha\boldsymbol{\xi}(\tau-P(\tau))$ in these contributions, the rotational invariance enforces them to be proportional to even powers of $\hbar$ times Feynman-Kac integrals, which also depend on $\hbar$ but are not completely canceled when $\hbar$ goes to zero (because the particle in the loop with $p=1$ is not involved in any exchange cyclic permutation). On the other hand, the exchange statistics generate contributions that vanish exponentially when $\hbar$ tends to zero because of the exchange part E_β^0 (3.14) of the internal energy of a loop with $p\geqslant 2$. [The factor $\exp(-\beta E_\beta^0)$ in the loop fugacity (3.5) survives in the loop density, after renormalization of the long-ranged divergencies of the Coulomb interactions, as shown in Sec. V.] For instance, at the order $|\mathbf{k}|^2$, the second term in curly brackets in (4.32) is equal to

$$-\sum_\alpha e_\alpha^2\sum_{p=1}^{\infty} p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})\int_0^p d\tau\,\lambda_\alpha\mathbf{k}\cdot\boldsymbol{\xi}_{P(\tau)+1}(\tau-P(\tau)) \times[\mathbf{k}\cdot\mathbf{X}(P(\tau))+\tfrac{1}{2}\lambda_\alpha\mathbf{k}\cdot\boldsymbol{\xi}_{P(\tau)+1}(\tau-P(\tau))]. \tag{4.33}$$

If $p=1$, $\mathbf{X}(P(\tau))=\mathbf{0}$ and the contribution from the loops with $p=1$ is proportional to $\lambda_\alpha^2\propto\hbar^2$ with a coefficient that remains finite when $\hbar$ goes to zero. On the other hand, the loops with $p\geqslant 2$ generate terms that may be proportional only to $\hbar$ but are exponentially canceled when $|\mathbf{k}|$ vanishes, as are all the terms with $p\geqslant 2$, because they involve $\rho_{\alpha,p}(\mathbf{X})$. A similar structure appears in the term of order $|\mathbf{k}|^2$ due to the loop correlations in the correction to $C^Q(\mathbf{k})$.

Usually, the induced charge density (4.32) is written in terms of the quantum static response function $\chi(\mathbf{r})\equiv\int_0^\infty dt\, C_{\text{ret}}^Q(\mathbf{r},t)$, where $C_{\text{ret}}^Q(\mathbf{r},t)$ is the quantum retarded charge-charge correlation function in real time t, which involves the average of the commutator of the charge density operators. Then, the static linear response reads $\Sigma_\alpha e_\alpha\rho_\alpha^{\text{ind}Q}(\mathbf{k})/\delta q(\mathbf{k})=-\beta v_C(\mathbf{k})C_{\text{ret}}^Q(\mathbf{k},\omega=0)$. This relation is valid at zero as well as finite temperature. At finite temperature, the retarded charge-charge correlation function $C_{\text{ret}}^Q(\mathbf{r},t)$ is related to the time-ordered charge-charge correlation function in imaginary time $C_T^Q(\mathbf{r},s)$. [s is the dimensionless real variable of Sec. III A ($0\leqslant s\leqslant 1$) and $C_T^Q(\mathbf{r},s)$ is a periodic function of s with period 1.] Indeed, by using the Lehman representation [36], it can be shown that the time Fourier transforms of both functions (over t and s, respectively) involve the same analytic function g: $C_{\text{ret}}^Q(\mathbf{k},\omega)=\lim_{\eta\to 0}g(\mathbf{k},\omega+i\eta)$, while $C_T(\mathbf{k},\nu_n)=g(\mathbf{k},i\nu_n)$, where n is a relative integer that indexes the discrete frequencies $\nu_n=2\pi n$. Consequently, the linear response can be expressed as

$$\frac{1}{\delta q(\mathbf{k})}\sum_\alpha e_\alpha\rho_\alpha^{\text{ind}Q}(\mathbf{k})=-\beta v_C(\mathbf{k})\int_0^1 ds\, C_T^Q(\mathbf{k},s). \tag{4.34}$$

On the other hand, the quantum static charge-charge correlation function is related to the quantum time-ordered charge-charge correlations by [36]

$$C^Q(\mathbf{k})=C_T^Q(\mathbf{k},s=0). \tag{4.35}$$

At a classical level, the linear response theory reads (4.25)

$$\frac{1}{\delta q(\mathbf{k})}\sum_{\alpha} e_\alpha \rho_\alpha^{\text{ind cl}}(\mathbf{k}) = -\beta v_C(\mathbf{k}) C^{\text{cl}}(\mathbf{k}), \tag{4.36}$$

where $C^{\text{cl}}(\mathbf{r})$ is defined by (4.30) with the classical values in place of the quantum ones. The classical equation (4.36) can be readily retrieved from the corresponding formula (4.32) of the loop formalism, because, in the classical limit, the shape $\mathbf{X}$ shrinks to zero and the correction terms inside the curly brackets of (4.32) vanish.

D. Coulombic screening

Screening rules analogous to (1.2) and (1.3) can be expressed in terms of the loop distributions via the linear response theory. The total induced charge $\int d\mathbf{r}\Sigma_\alpha e_\alpha \rho_\alpha^{\text{ind}Q}(\mathbf{r})$ is finite and since the correction terms to $C^Q(\mathbf{k})$ in the linear response equation (4.32) vanish when $\mathbf{k}=\mathbf{0}$, this implies that

$$\int d\mathbf{r}\; C^Q(\mathbf{r}) = 0. \tag{4.37}$$

Equation (4.37) is a consequence of (1.2): the total charge of the polarization cloud around the charge $\Sigma_\alpha e_\alpha \rho_\alpha(\mathbf{r}) d\mathbf{r}$ contained in an infinitesimal volume $d\mathbf{r}$ of the medium is exactly opposite to this charge. The property (4.37) can also be derived from the imaginary-time equations of motion, under the assumption that the correlation functions have an inverse-power asymptotic expansion starting as $1/r^3$ at large distances [11]. According to the definitions (4.5) and (4.30), the screening equation (4.37) implies that

$$\sum_\alpha e_\alpha^2 \left[\rho_\alpha^Q + \int d\mathbf{r}\; \rho_{\alpha\alpha}^{(2)TQ}\Big|_{\text{exch}}(\mathbf{r}) \right] = -\sum_\alpha \sum_{\alpha'} e_\alpha e_{\alpha'} \int d\mathbf{r}\; \rho_{\alpha\alpha'}^{(2)TQ}|_{\text{nonexch}}(\mathbf{r}). \tag{4.38}$$

Charges that are not exchanged are expected to attract or repel each other according to whether they have opposite signs or not. Henceforth, in the polarization cloud around a charge e_α of the medium, the total charge of the part of the cloud that is build by particles that are not exchanged with e_α, namely, $\Sigma_{\alpha'} e_{\alpha'} \int d\mathbf{r}[\rho_{\alpha\alpha'}^{(2)TQ}|_{\text{nonexch}}(\mathbf{r})/\rho_\alpha]$ has a sign opposite to e_α, so that, according to the identity (4.9),

$$\sum_\alpha e_\alpha^2 \sum_{p=1}^{\infty} p^2 \int D(\mathbf{X}) \rho_{\alpha,p}(\mathbf{X}) = \sum_\alpha e_\alpha^2 \left[\rho_\alpha^Q + \int d\mathbf{r}\; \rho_{\alpha\alpha}^{(2)TQ}\Big|_{\text{exch}}(\mathbf{r}) \right] > 0. \tag{4.39}$$

Equation (4.39) is valid even in the case of fermions, for which the sign of $\rho_{\alpha,p}(\mathbf{X})$ is expected to depend on p. The inequality (4.39) will be used in Sec. V.

Moreover, since Coulomb systems in dimension 3 can be assumed to be in a conductive phase, the total induced charge in the bulk is expected to be exactly equal to the opposite of the infinitesimal external charge $\int d\mathbf{r}\; \delta q(\mathbf{r})$ (the excess charge being confined near the boundaries of the system). According to (4.32), the corresponding equation (1.3) implies that the second moment of the quantum charge-charge correlation function satisfies the equation

$$\begin{aligned} \frac{1}{6}\int d\mathbf{r}\; r^2 C^Q(\mathbf{r}) &= -\frac{1}{4\pi\beta} - \frac{1}{6}\Bigg\{ \sum_\alpha e_\alpha^2 \sum_{p=1}^{\infty} p \int D_p(\mathbf{X}) \rho_{\alpha,p}(\mathbf{X}) \\ &\quad \times \int_0^p d\tau \{[\mathbf{X}(\tau)]^2 - [\mathbf{X}(P(\tau))]^2\} \\ &\quad + \sum_\alpha e_\alpha \sum_{\alpha'} e_{\alpha'} \sum_{p=1}^{\infty} p \sum_{p'=1}^{\infty} \\ &\quad \times \int d\mathbf{r} \int D(\mathbf{X}) \int D(\mathbf{X}') \rho_{\alpha,p;\alpha',p'}^{(2)T}(\mathbf{r},\mathbf{X},\mathbf{X}') \\ &\quad \times \int_0^{p'} d\tau' \{[\mathbf{r}+\mathbf{X}'(\tau')]^2 - \mathbf{r}^2\} \Bigg\}. \end{aligned} \tag{4.40}$$

Equation (4.40) is the version of the screening sum rule (1.3) written in the loop formalism. Usually (1.3) is written in terms of the retarded charge-charge correlation function in imaginary time $s, C_T(\mathbf{r},s)$, as

$$\frac{1}{6}\int d\mathbf{r}\; r^2 \int_0^1 ds\; C_T(\mathbf{r},s) = -\frac{1}{4\pi\beta}. \tag{4.41}$$

This sum rule can also be derived from the equilibrium equation under the same assumptions as for (4.37) [11]. In the classical limit, the shape $\mathbf{X}$ shrinks to zero and (4.40) tends to the well-known Stillinger-Lovett sum rule [10]

$$\frac{1}{6}\int d\mathbf{r}\; r^2 C^{\text{cl}}(\mathbf{r}) = -\frac{1}{4\pi\beta}. \tag{4.42}$$

In Fourier representation, this sum rule reads $\lim_{|\mathbf{k}|\to 0} \beta v_C(\mathbf{k}) C^{\text{cl}}(\mathbf{k}) = 1$ and it can be derived by inserting the classical linear response (4.36) in (1.3).

For the OCP, the charge density is proportional to the particle density and the position of the mass center of moving particles is only subjected to the harmonic force due to the charged background. Thus it can be shown exactly [30,37] that the second moment of the quantum charge-charge correlation does not obey the classical sum rule (4.42) since it satisfies

$$\frac{1}{6}\int d\mathbf{r}\; r^2 C_{\text{OCP}}^Q(\mathbf{r}) = -\frac{1}{4\pi\beta} \frac{\beta\hbar\omega_P}{2} \coth\left(\frac{\beta\hbar\omega_P}{2}\right), \tag{4.43}$$

where ω_P is the plasma frequency $\omega_P = (4\pi\rho e^2/m)^{1/2}$.

V. RESUMMATION OF THE COULOMB DIVERGENCIES IN THE LOOP VIRIAL EXPANSIONS

A. Virial expansions for the classical gas of loops

The quantum particle-particle correlation has contributions from both the loop density and the loop correlations. Since the classical loops interact through a two-body potential $v(\mathscr{L}_i,\mathscr{L}_j)$, usual techniques of classical statistics can be applied to the system of loops, which was done by Ginibre in the case of short-range potentials [1,17,38]. The notion of standard Mayer bonds, which were originally introduced for point objects [14], has already been generalized to extended objects for the Coulomb potential [21,25] in formalisms where the exchange effects were treated perturbatively. In the present paper, we introduce Mayer bonds for the classical extended exchange loops by setting

$$f(\mathscr{L}_i,\mathscr{L}_j)=e^{-\beta_{ij}v(\mathscr{L}_i,\mathscr{L}_j)}-1, \tag{5.1}$$

with $\beta_{ij}\equiv\beta e_{\alpha_i}e_{\alpha_j}$. The usual virial diagrammatics can be applied to the system of loops and we call ''points'' of the diagrams the loop objects $\mathscr{L}$.

The present section can be summed up as follows. In Sec. V A we introduce the virial diagrams for the loop-fugacity expansion of the density of loops and for the loop-density expansion of the Ursell function of loops. The loop-fugacity expansion of the density of loops allows one to study the part of the particle-particle correlation that comes from configurations where the two particles considered are exchanged within the same cycle; because of the strong connectivity of the loop-density diagrams (the absence of articulation points), the loop-density expansion of the Ursell function is more adequate than its loop-fugacity expansion to study the large-distance behavior of the part of the quantum correlations induced by the loop correlations. Every Mayer diagram diverges because of the nonintegrability of the $1/r$ decay of the loop potential at large distances. The exact resummation process displayed by Meeron [2] for the classical Coulomb gas is generalized to the system of loops. The resummation scheme for the loop-fugacity expansion of the loop density is displayed in Appendix B, while the analogous and simpler resummation for the loop-density expansion of the Ursell-function is performed in Sec. V B as follows. By using the multipolar decomposition of the potential introduced in Sec. III C, the initial Mayer bond is decomposed into auxiliary bonds $\tilde{f}$ associated with auxiliary diagrams $\tilde{\Gamma}$. The $\tilde{f}$ bonds describe, respectively, the charge-charge interaction (f^{cc}), the multipole-charge interactions (f^{mc} and f^{cm}), the multipole-multipole interaction (f^{mm}), and the rest of the f bond (f_T). In order to resum the Coulomb divergencies involving the total charges of the loops, the $\tilde{\Gamma}$ diagrams are gathered into equivalence classes characterized by a prototype diagram, where the so-called Coulomb chains of charge-charge bonds ending with either an f^{cc} bond or an f^{mc} bond are suppressed. The definition of the prototype diagrams with several kinds of F bonds together with the associated topological rules ensures the existence of a one-to-one correspondence between the partition of the auxiliary diagrams and the set of the prototype diagrams. The intermediate points of the Coulomb chains are integrated first and the resulting resummation between two points of a prototype diagram is thoroughly independent from the resummation between the points of another pair. In Sec. V C the explicit values of the resummed bonds F are calculated. The so-called F^{cc}, F^{mc}, and F^{cm} bonds decay exponentially at large distances, whereas the fourth bond F_R behaves as a kind of dipole-dipole interaction: the charge-charge and multipole-charge interaction are classically screened, but the multipole-multipole interaction, which is typically quantum, induces algebraic tails. The integrability of the resummed prototype diagrams for the loop-fugacity as well as the loop-density expansions is discussed in Appendix C. In the case of the loop-fugacity expansion, the integrability is ensured by performing first the integration over the shapes of the loops, and it is discussed how the loop density decays faster than any inverse power of the distance between two points in the loop. Then the integrability of the diagrams in the loop-density expansion of the Ursell function is derived from the connectivity of the diagrams. In Sec. V D the part of the particle-particle correlation arising directly from exchange effects (i.e., the part that is derived from the loop density) is bounded by a series, which is expected to decay faster than any inverse power law, as the series corresponding to the particle-particle correlation in a noninteracting system.

The loop-fugacity expansion of the density of loops has the diagrammatic representation [13,14]

$$\rho(\mathscr{L}_a)=z(\mathscr{L}_a)\sum_{\mathbb{G}}\frac{1}{S_{\mathbb{G}}}\int\prod_{n=1}^{N}[d\mathscr{L}_n z(\mathscr{L}_n)]\Big[\prod f\Big]_{\mathbb{G}}. \tag{5.2}$$

In (5.2) the sum runs over all the unlabeled topologically different connected diagrams $\mathbb{G}$ with one root point $\mathscr{L}_a$ (which is not integrated over) and N internal points ($N=0,\ldots,\infty$). Each pair of points in $\mathbb{G}$ is linked by at most one f bond and every point has a weight equal to 1. $[\Pi f]_{\mathbb{G}}$ is the product of the f bonds in the $\mathbb{G}$ diagram and $S_{\mathbb{G}}$ is its symmetry factor, i.e., the number of permutations of the internal points $\mathscr{L}_n$ that do not change this product. For brevity, we have used the convention that, if N is equal to 0, no $(1/S_{\mathbb{G}})\int\Pi_{n=1}^{N}[d\mathscr{L}_n(\mathscr{L}_n)][\Pi f]_{\mathbb{G}}$ appears and the corresponding contribution to $\rho(\mathscr{L}_a)$ reduces to $z(\mathscr{L}_a)$.

The truncated two-loop distribution is related to the two-body Ursell function $h(\mathscr{L}_a,\mathscr{L}_b)$ of the loops via

$$\rho^{(2)T}(\mathscr{L}_a,\mathscr{L}_b)=\rho(\mathscr{L}_a)\rho(\mathscr{L}_b)h(\mathscr{L}_\alpha,\mathscr{L}_b). \tag{5.3}$$

The topological structure of the virial diagrammatics implies that the Ursell function has the simple loop-density expansion

$$h(\mathscr{L}_a,\mathscr{L}_b)=\sum_{\Gamma}\frac{1}{S_{\Gamma}}\int\prod_{n=1}^{N}[d\mathscr{L}_n\rho(\mathscr{L}_n)]\Big[\prod f\Big]_{\Gamma}. \tag{5.4}$$

In (5.4) the sum runs over all the unlabeled topologically different connected diagrams Γ with two root points $\mathscr{L}_a$ and $\mathscr{L}_b$ and N internal points ($N=0,\ldots,\infty$) that are built as the $\mathbb{G}$ diagrams, apart from the fact that a Γ diagram contains no ''articulation'' point. An articulation point is defined by the fact that, if it is taken out of the diagram, then the latter is split into two pieces, one of which at least is no longer linked

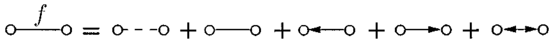

FIG. 2. Diagrammatic representation of the decomposition (5.6) of an f bond (between two points $\mathscr{L}_i$ and $\mathscr{L}_j$ denoted by two white circles) into five auxiliary $\tilde{f}$ bonds. A dotted line is an f_T bond, a solid line an f^{cc} bond, a solid line between $\mathscr{L}_i$ and $\mathscr{L}_j$ with one arrow pointing to the point $\mathscr{L}_i$ a bond $f^{mc}(\mathscr{L}_i,\mathscr{L}_j)$, and a solid line with arrows at both ends a f^{mm} bond.

to any root point. We mention that, if N is equal to 0, no $\int d\mathscr{L}_n\rho(\mathscr{L}_n)$ appears and $(1/S_\Gamma)[\Pi f]_\Gamma$ is reduced to $f(\mathscr{L}_a,\mathscr{L}_b)$. The absence of articulation points is the only difference between the topology of the diagrams in the density expansion and the topology of the corresponding diagrams in the fugacity expansion. The connectivity is ''stronger'' in the first case and, subsequently, the loop-density expansion is more adequate for the discussion of the large-distance behavior of the Ursell function.

When the distance R_{ij} between the positions of the loops goes to infinity, the pair potential (3.13) between the loops decreases as the Coulombic potential $1/R_{ij}$. The nth term in the Taylor expansion (3.20) of the potential $v(\mathbf{R}_{ij},\mathbf{X}_i,\mathbf{X}_j)$ decreases as $1/R_{ij}^{n+1}$, so the dominant asymptotic behavior of the Mayer bond $f(\mathscr{L}_i,\mathscr{L}_j)$ defined in (5.1) is

$$f^{cc}(\mathscr{L}_i,\mathscr{L}_j)\equiv-\beta(p_ie_{\alpha_i})(p_je_{\alpha_j})v_C(R_{ij})=-\beta_{ij}v^{cc}(c_i,c_j), \tag{5.5}$$

with the notations of (3.21). So the integrals corresponding either to the $\mathbb{G}$ diagrams of the expansion (5.2) or to the Γ diagrams of the expansion (5.4) diverge.

B. Topological principles of the resummation

These large-distance Coulomb divergencies are dealt with by means of an exact partial resummation of auxiliary diagrams, according to a method introduced by Meeron in the classical case [2]. The resummation scheme for the loop-fugacity expansion of the density of loops $\rho(\mathscr{L})$ follows the same lines as that for the loop-density expansion of the Ursell function $h(\mathscr{L},\mathscr{L}')$. However, the topological principles of the resummation are simpler in the loop-density expansion than in the loop-fugacity expansion and the latter is displayed in Appendix B. In order to exhibit the part of the pair potential that is classically screened, we use the ''multipolar'' decomposition (3.22) of the loop potential. We split the original bond $f(\mathscr{L}_i,\mathscr{L}_j)$ defined in (5.1) into the sum (see Fig. 2)

$$f=f_T(\mathscr{L}_i,\mathscr{L}_j)+f^{cc}(c_i,c_j)+f^{mc}(\mathscr{L}_i,c_j)+f^{cm}(c_i,\mathscr{L}_j)+f^{mm}(\mathscr{L}_i,\mathscr{L}_j). \tag{5.6}$$

The truncated bond f_T

$$f_T(\mathscr{L}_i,\mathscr{L}_j)\equiv e^{-\beta_{ij}v(\mathscr{L}_i,\mathscr{L}_j)}-1+\beta_{ij}v(\mathscr{L}_i,\mathscr{L}_j) \tag{5.7}$$

decreases only as $[\beta(p_ie_{\alpha_i})(p_je_{\alpha_j})]^2/(2R_{ij}^2)$, while the rest of (5.6) is inspired by the multipolar decomposition (3.22) of $v(\mathscr{L}_i,\mathscr{L}_j)$ rewritten as

$$-\beta_{ij}v=f^{cc}+f^{mc}+f^{cm}+f^{mm}, \tag{5.8}$$

with $f^{mc}(\mathscr{L}_i,c_j)\equiv-\beta_{ij}v^{mc}(\mathscr{L}_i,c_j)$, $f^{cm}(c_i,\mathscr{L}_j)\equiv-\beta_{ij}v^{cm}(c_i,\mathscr{L}_j)$, and $f^{mm}(\mathscr{L}_i,\mathscr{L}_j)\equiv-\beta_{ij}v^{mm}(\mathscr{L}_i,\mathscr{L}_j)$. With the decomposition of the bond f into the sum (5.6), $h(\mathscr{L}_a,\mathscr{L}_b)$ can be expressed by the formula (5.4), where now the Γ diagrams are replaced by $\tilde{\Gamma}$ diagrams made with $\tilde{f}$ bonds that are equal either to f_T, f^{cc}, f^{mc}, f^{cm}, or f^{mm},

$$h(\mathscr{L}_a,\mathscr{L}_b)=\sum_{\tilde{\Gamma}}\frac{1}{S_{\tilde{\Gamma}}}\int\prod_{n=1}^{N}[d\mathscr{L}_n\rho(\mathscr{L}_n)]\Big[\prod\tilde{f}\Big]_{\tilde{\Gamma}}. \tag{5.9}$$

The Γ and $\tilde{\Gamma}$ diagrams have the same topological properties because the f bond is just the sum of the various $\tilde{f}$ bonds. An example of a $\tilde{\Gamma}$ diagram is shown in Fig. 3(a).

Our purpose is to resum all the Coulombic divergencies involving the total charge of a loop $\mathscr{L}_n$, i.e., the total charge of the corresponding p_n quantum particles. In order to achieve this aim, we first integrate over all the intermediate

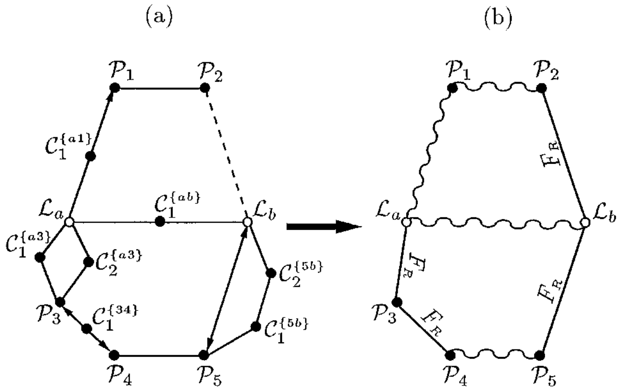

FIG. 3. (a) Typical auxiliary $\tilde{\Gamma}$ diagram and (b) the prototype Π diagram to which $\tilde{\Gamma}$ contributes by the resummation process of Sec. V B. The white circles are root points and the black circles are the internal points that are integrated over with the measure $\int d\mathscr{L}\rho(\mathscr{L})$ in (5.9). The notations for the bonds in $\tilde{\Gamma}$ are those of Fig. 2. In the corresponding Π diagram, the Coulomb points $\mathscr{C}_k^{\{ij\}}$ defined in Sec. V B are suppressed and the points $\mathscr{P}_i$ that are left over are linked by F bonds. An F^{cc} bond is represented by a wavy line, a bond $F^{mc}(\mathscr{P}_i,\mathscr{P}_j)$ by a wavy line with an arrow pointing to $\mathscr{P}_i$, and an F_R bond is denoted by a solid line with a superscript F_R.

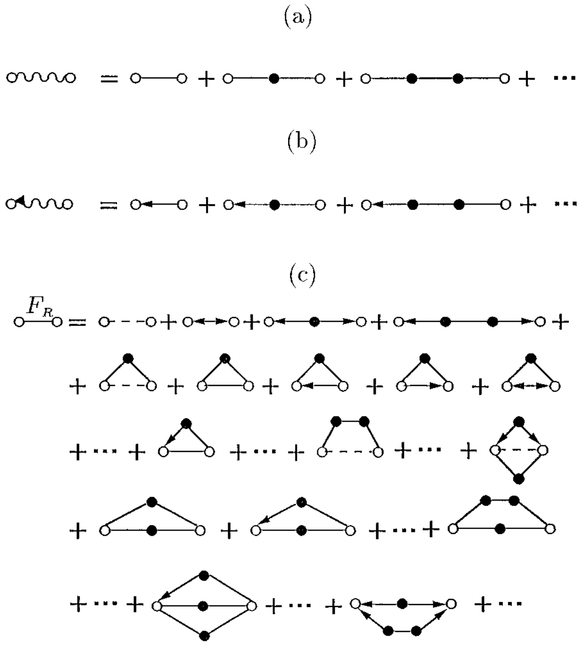

FIG. 4. Diagrammatic representation of the resummed F bonds, according to the definitions of Sec. V B. With the notations of Fig. 3, (a) corresponds to an F^{cc} bond, (b) to an F^{mc} bond, and (c) to an F_R bond.

points of the convolutions $f^{cc}*f^{cc}$, $f^{mc}*f^{cc}$, $f^{cc}*f^{cm}$, and $f^{mc}*f^{cm}$. In the following, these intermediate points are called ''(convolution) Coulomb points'' $\mathscr{C}_k$. The convolution chains that contain only intermediate Coulomb points are referred to as ''Coulomb chains''; a Coulomb chain without any internal point is either an f^{cc}, f^{mc}, f^{cm}, or f^{mm} bond. First, we notice that we can make a partition of the $\tilde{\Gamma}$ diagrams such that all the diagrams in a given class lead to the same so-called prototype Π diagram, when all the Coulomb points are supressed (in the sense that they are already integrated over). An example is given in Fig. 3(b). There is at most one link between any two points that are left over in a Π diagram through this process. As shown in Ref. [25], mere combinatory and topological considerations lead to the identity

$$h(\mathscr{L}_a,\mathscr{L}_b)=\sum_{\Pi}\frac{1}{S_{\Pi}}\int\prod_{m=1}^{M}[d\mathscr{P}_m\rho(\mathscr{P}m)]\Big[\prod F\Big]_{\Pi}, \tag{5.10}$$

with the definition

$$F(\mathscr{P}_i,\mathscr{P}_j)=\sum_{\tilde{\Gamma}_{ij}}\frac{1}{S_{\tilde{\Gamma}_{ij}}}\int\prod_{k=1}^{n_{ij}}[d\mathscr{C}_k^{\{ij\}}\rho(\mathscr{C}_k^{\{ij\}})]\Big[\prod\tilde{f}\Big]_{\tilde{\Gamma}_{ij}}. \tag{5.11}$$

In (5.11) the sum runs over all the unlabeled $\tilde{\Gamma}_{ij}$ diagrams that are built between the two root points $\mathscr{P}_i$ and $\mathscr{P}_j$ by adding Coulomb chains made with n_{ij} Coulomb points $\mathscr{C}_k^{\{ij\}}$, according to some prescriptions explained below.

The identity (5.10) is valid only if the definition of the Π diagrams is such that there does exist a one-to-one correspondence between the partition of the $\tilde{\Gamma}$ diagrams and the set of the Π diagrams. In fact, in the process that associates a given $\tilde{\Gamma}$ diagram with a Π diagram, a point $\mathscr{L}_n$ that is connected to at least three other points in $\tilde{\Gamma}$ is kept as a point $\mathscr{P}_m$ in Π, whereas an intermediate point in a convolution of bonds in $\tilde{\Gamma}$ disappears if and only if it is a Coulomb point. Thus we have to introduce various kinds of F bonds that can be distinguished from one another by some corresponding excluded-convolution rules. We call the F bond that links two points $\mathscr{P}_i$ and $\mathscr{P}_j$ of a Π diagram

(i) a ''charge-charge'' bond $F^{cc}(\mathscr{P}_i,\mathscr{P}_j)$ if $\mathscr{P}_i$ and $\mathscr{P}_j$ are linked in the $\tilde{\Gamma}$ diagram by a single chain of f^{cc} bonds [see Fig. 4(a)],

(ii) a ''multipole-charge'' bond $F^{mc}(\mathscr{P}_i,\mathscr{P}_j)$ [a ''charge-multipole'' bond $F^{cm}(\mathscr{P}_i,\mathscr{P}_j)$] if the points are linked in the $\tilde{\Gamma}$ diagram by a single chain made with an f^{mc} bond starting at $\mathscr{P}_i$ convoluted with a chain of f^{cc} bonds ending at $\mathscr{P}_j$ [a single chain of f^{cc} bonds starting at $\mathscr{L}_i$ convoluted with an f^{cm} bond ending at $\mathscr{P}_j$] [see Fig. 4(b)], and

(iii) a ''dressed'' F_R bond in other cases [see Fig. 4(c)]. Then the excluded-convolution rules that distinguish the F bonds read as follows: in the Π diagram there cannot be any convolution $F^{cc}*F^{cc}$, $F^{mc}*F^{cc}$, $F^{cc}*F^{cm}$, or $F^{mc}*F^{cm}$. This rule is exemplified in Fig. 3. We stress that the resummations of Coulomb chains between the various pairs of points $\{\mathscr{P}_i,\mathscr{P}_j\}$ are thoroughly independent from one another. Hence the topological structure of the Mayer $\tilde{\Gamma}$ diagrams is preserved through the resummation process, apart from the above extra excluded-convolution rules, which avoid double counting in the correspondence between $\tilde{\Gamma}$ and Π diagrams.

C. Explicit values of the resummed bonds

1. The screened charge-charge bond F^{cc}

According to the definition of the preceding subsection,

$$\begin{aligned}F^{cc}(\mathscr{P}_i,\mathscr{P}_j)={}&f^{cc}(\mathscr{P}_i,\mathscr{P}_j)\\&+\sum_{N=1}^{\infty}\int\Bigg[\prod_{k=1}^{N}d\mathscr{C}_k^{\{ij\}}\rho(\mathscr{C}_k^{\{ij\}})\Bigg]f^{cc}(\mathscr{P}_i,\mathscr{C}_1^{\{ij\}})\\&\times f^{cc}(\mathscr{C}_1^{\{ij\}},\mathscr{C}_2^{\{ij\}})\cdots f^{cc}(\mathscr{C}_N^{\{ij\}},\mathscr{P}_j).\end{aligned} \tag{5.12}$$

Since the f^{cc} bonds (5.5) do not depend on the shapes of the loops, the integration over the internal degrees of freedom $(\alpha,p,\mathbf{X})$ of the N intermediate Coulomb loops $\mathscr{C}_k^{\{ij\}}$ in a convolution chain of $N+1 f^{cc}$ bonds factors out as $[\Sigma_{\alpha=1}^{n_s}\Sigma_{p=1}^{\infty}\int D(\mathbf{X})e_{\alpha}^2p^2\rho_{\alpha,p}(\mathbf{X})]^N$.

The contribution F^{cc} from the convolution chains of f^{cc} bonds with all possible lengths (i.e., all possible numbers N of intermediate points) is readily calculated by a Fourier transform over the positions $\mathbf{R}_i$ of the loops, as in the diagrammatic version of the Debye-Hückel theory [15]. With the notation $g(\mathbf{k})\equiv\int d\mathbf{r}\,\exp(i\mathbf{k}\cdot\mathbf{r})g(\mathbf{r})$,

$$\int d\mathbf{R}_{ij} e^{i\mathbf{k}\cdot\mathbf{R}_{ij}} F^{cc}(\mathscr{L}_i,\mathscr{L}_j)$$

$$= -\beta(p_i e_{\alpha_i})(p_j e_{\alpha_j})\left\{ v_C(\mathbf{k}) + \sum_{N=1}^{\infty}\left(-\beta \sum_{\alpha=1}^{n_s} e_\alpha^2 \sum_{p=1}^{\infty} p^2 \times \int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})\right)^N [v_C(\mathbf{k})]^N \right\}$$

$$= -\beta_{ij} p_i p_j \frac{4\pi}{\mathbf{k}^2+\kappa^2}, \tag{5.13}$$

where $\kappa^2 = 4\pi\beta\Sigma_\alpha e_\alpha^2 \Sigma_{p=1}^{\infty} p^2 \int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})$. According to (4.9),

$$\kappa^2 = 4\pi\beta \sum_\alpha e_\alpha^2 \left[\rho_\alpha^Q + \int d\mathbf{r}\, \rho_{\alpha\alpha}^{(2)TQ}\Big|_{\text{exch}}(\mathbf{r}) \right]. \tag{5.14}$$

κ^2 is positive and finite, according to (4.39). Thus κ is real and finite and the resummed charge-charge bond F^{cc} reads

$$F^{cc}(c_i,c_j) = -\beta_{ij} p_i p_j \phi(r_{ij}) = -\beta_{ij}\phi^{cc}(c_i,c_j), \tag{5.15}$$

where $\phi^{cc}(c_i,c_j) = p_i p_j \phi(r_{ij})$ and ϕ is a potential in the manner of Debye

$$\phi(r) = \frac{e^{-\kappa r}}{r}. \tag{5.16}$$

In the fermionic quantum regime at high density, the kinetic energy becomes far larger than the interaction energy and the correlations due to the interactions become negligible; then, according to (4.9), $\rho_\alpha^Q + \int d\mathbf{r}\, \rho_{\alpha,\alpha}^{(2)TQ}|_{\text{exch}}(\mathbf{r})$ becomes equal to $\partial\rho_\alpha^{0Q}/\partial(\beta\mu_\alpha)$ and the value (5.14) of κ^2 tends to the random-phase approximation (RPA) expression $\kappa_{\text{RPA}}^2 = 4\pi\beta\Sigma_\alpha e_\alpha^2 \partial\rho_\alpha^{0Q}/\partial(\beta\mu_\alpha)$. In the classical limit, $\rho_{\alpha\alpha}^{(2)TQ}|_{\text{exch}}$ vanishes and we retrieve the inverse Debye-Hückel screening length $\kappa_{\text{DH}} = [4\pi\beta\Sigma_\alpha e_\alpha^2 \rho_\alpha]^{1/2}$. A more detailed comparison with the screening lengths of the usual mean-field theories and of the RPA theory will be given in Sec. V of paper II.

2. The screened multipole-charge bond F^{mc}

The only difference between an F^{mc} bond and an F^{cc} bond is that there is an end bond f^{mc} (5.8) in place of a bond f^{cc} starting at point $\mathscr{P}_i$. By means of the relation

$$[(\mathbf{X}_i\cdot\nabla_i)^q f^{cc}] * f^{cc} * \cdots * f^{cc} = (\mathbf{X}_i\cdot\nabla_i)^q [f^{cc} * f^{cc} * \cdots * f^{cc}], \tag{5.17}$$

the calculation of the F^{mc} bond can be deduced directly from that of F^{cc}. The result is

$$F^{mc}(\mathscr{L}_i,c_j) = -\beta_{ij} p_j \int_0^{p_i} d\tau \sum_{q=1}^{\infty} \frac{1}{q!} [\mathbf{X}_i(\tau)\cdot\nabla_i]^q \phi(R_{ij})$$

$$= -\beta_{ij}[\phi_{\text{elect}}(\mathscr{L}_i,c_j) - \phi^{cc}(c_i,c_j)], \tag{5.18}$$

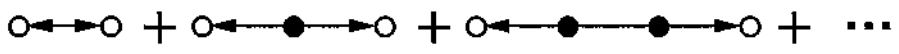

FIG. 5. Diagrammatic representation of the F^{mm} bond defined in Sec. V C, which is just an auxiliary object in the calculation of the F_R bond.

where $\phi_{\text{elect}}(\mathscr{L}_i,c_j)$ is the electrostatic ‘‘Debye-Hückel’’ potential between a point charge p_j and a charged curve $\mathbf{\Omega}_i$ with a shape $\mathbf{X}_i$ and a charge density $\sigma_i(\mathbf{r}) = \int_0^{p_i} d\tau\, \delta(\mathbf{r} - \mathbf{\Omega}_i(\tau))$ [see (3.23)],

$$\phi_{\text{elect}}(\mathscr{L}_i,c_j) = p_j \int_0^{p_i} d\tau\, \phi[\mathbf{\Omega}_i(\tau) - \mathbf{R}_j].$$

In a similar way, we obtain

$$F^{cm}(c_i,\mathscr{L}_j) = -\beta_{ij}[\phi_{\text{elect}}(c_i,\mathscr{L}_j) - \phi^{cc}(c_i,c_j)]. \tag{5.19}$$

3. The screened dressed bond F_R

$F_R(\mathscr{P}_i,\mathscr{P}_j)$ is the sum of all the subdiagrams $\tilde{\Gamma}_{ij}$ that appear in (5.11) and are not included in F^{cc}, F^{mc}, or F^{cm} because they can be convoluted with any other subdiagram $\tilde{\Gamma}_{ki}$ or $\tilde{\Gamma}_{jl}$ [see Fig. 4(c)]. F_R can be viewed as the sum of two kinds of contributions. The first one involves the single bonds f_T and F^{mm}, where F^{mm} denotes the sum of the single chains with all possible lengths in which an f^{mc} bond starting at $\mathscr{P}_i$ is convoluted with a possible chain of f^{cc} bonds that is itself convoluted with an f^{cm} bond ending at $\mathscr{P}_j$ (see Fig. 5). The second kind of contribution corresponds to multiple chains. According to the definition (5.9) of the $\tilde{\Gamma}$ diagrams, there is at most one $\tilde{f}$ bond between two points of those diagrams and we have to introduce the notion of ‘‘genuine’’ Coulomb chain, i.e., a Coulomb chain that contains at least one intermediate Coulomb point. Then the multiple chains that contribute to F_R can be expressed either as the product of one among the five $\tilde{f}$ bonds with n genuine Coulomb chains with $n \geqslant 1$ or the product of n genuine Coulomb chains with $n \geqslant 2$. As explicitly recalled in Ref. [25], the symmetry factors of the products of similar chains are such that, when the products of n genuine Coulomb chains are summed over all possible chain lengths, the result is merely equal to $(-\beta_{ij}\psi_{\text{chain}})^n/n!$, where $-\beta_{ij}\psi_{\text{chain}}$ denotes the sum of the genuine Coulomb chains with all possible lengths. Eventually, since f is the sum of the various $\tilde{f}$, F_R [see Fig. 4(c)] can be written as

$$F_R = f_T + F^{mm} + f[e^{-\beta_{ij}\psi_{\text{chain}}} - 1] + [e^{-\beta_{ij}\psi_{\text{chain}}} - 1 + \beta_{ij}\psi_{\text{chain}}]. \tag{5.20}$$

According to the relation (3.22) together with the definitions of F^{cc}, F^{mc}, F^{cm}, and F^{mm}, the value of ψ_{chain} can be deduced from the relation

$$-\beta_{ij}[v + \psi_{\text{chain}}] = F^{cc} + F^{mc} + F^{cm} + F^{mm}. \tag{5.21}$$

By using this relation together with the definitions (5.1) and (5.7), (5.20) can be rewritten as

$$F_R = e^{F^{cc}+F^{mc}+F^{cm}+F^{mm}} - 1 - F^{cc} - F^{mc} - F^{cm}. \tag{5.22}$$

Only the calculation of the explicit value of F^{mm} is left to be performed.

F^{mm} is the sum that involves, on one hand, the basic bond $f^{mm}(\mathscr{P}_i,\mathscr{P}_j)$, where $\tau - P(\tau)$ must be equal to $\tau' - P(\tau')$, and, on the other hand, the genuine Coulomb chains in which an f^{mc} bond with a parameter τ starts at $\mathscr{P}_i$ and an f^{cm} bond with any other parameter τ' ends at $\mathscr{P}_i$. According to the identity

$$\begin{aligned}&[(\mathbf{X}_i\cdot\boldsymbol{\nabla}_i)^q f^{cc}]*f^{cc}*\cdots*f^{cc}*[(\mathbf{X}_j\cdot\boldsymbol{\nabla}_j)^{q'} f^{cc}]\\ &\quad=(\mathbf{X}_i\cdot\boldsymbol{\nabla}_i)^q(\mathbf{X}_j\cdot\boldsymbol{\nabla}_j)^{q'}[f^{cc}*f^{cc}*\cdots*f^{cc}*f^{cc}],\end{aligned} \tag{5.23}$$

the value of F^{mm} is calculated as that of F^{cc}, with two bonds f^{cm} (5.8) at the end points,

$$\begin{aligned}F^{mm} = &-\beta_{ij}\int_0^{p_i} d\tau \int_0^{p_j} d\tau' \sum_{q=1}^{\infty}\sum_{q'=1}^{\infty}\frac{1}{q!}\frac{1}{q'!}\\ &\times[\mathbf{X}_i(\tau)\cdot\boldsymbol{\nabla}_i]^q[\mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_j]^{q'}\\ &\times\{\delta([\tau - P(\tau)] - [\tau' - P(\tau')])v_C(R_{ij})\\ &+[\phi(R_{ij}) - v_C(R_{ij})]\}.\end{aligned} \tag{5.24}$$

According to (3.22) and (3.28), the pure Coulombic part in F^{mm} can be written as

$$\begin{aligned}W(\mathscr{L}_i,\mathscr{L}_j) &= -\beta_{ij}[v(\mathscr{L}_i,\mathscr{L}_j) - v_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j)]\\ &= -\beta_{ij}\int_0^{p_i} d\tau \int_0^{p_j} d\tau'\\ &\quad\times\{\delta([\tau_i - P(\tau_i)] - [\tau_i' - P(\tau_i')]) - 1\}\\ &\quad\times v_C[\boldsymbol{\Omega}_i(\tau) - \boldsymbol{\Omega}_i(\tau')],\end{aligned} \tag{5.25}$$

where $v_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j)$ is the electrostatic potential between two classical loops defined in (3.27). The corresponding screened Debye potential is

$$\phi_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j) = \int_0^{p_i} d\tau \int_0^{p_j} d\tau'\, \phi[\boldsymbol{\Omega}_i(\tau) - \boldsymbol{\Omega}_j(\tau')] \tag{5.26}$$

and, by using (5.15), (5.18), and (5.19), the Debye part in the expression (5.24) of F^{mm} can be rewritten in terms of this potential as

$$\begin{aligned}&-\beta_{ij}\phi_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j) - F^{cc}(c_i,c_j) - F^{mc}(\mathscr{L}_i,c_j)\\ &\qquad - F^{cm}(c_i,\mathscr{L}_j).\end{aligned} \tag{5.27}$$

Eventually, by collecting (5.25) and (5.27), we get the relation

$$\begin{aligned}&F^{cc}(c_i,c_j) + F^{mc}(\mathscr{L}_i,c_j) + F^{cm}(c_i,\mathscr{L}_j) + F^{mm}(\mathscr{L}_i,\mathscr{L}_j)\\ &\quad= -\beta_{ij}[v(\mathscr{L}_i,\mathscr{L}_j) - v_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j) + \phi_{\text{elect}}(\mathscr{L}_i,\mathscr{L}_j)].\end{aligned} \tag{5.28}$$

We notice that the value of ψ_{chain} is derived by comparison with (5.21),

$$\psi_{\text{chain}} = \phi_{\text{elect}} - v_{\text{elect}}, \tag{5.29}$$

and that it vanishes when the density goes to zero.

The explicit value of F_R is derived from (5.22) and (5.28),

$$F_R = e^{-\beta_{ij}[v+\phi_{\text{elect}} - v_{\text{elect}}]} - 1 - F^{cc} - F^{mc} - F^{cm}. \tag{5.30}$$

The dressed bond F_R depends on the density only through the inverse length κ. Since ϕ, and consequently ϕ_{elect}, decreases exponentially at large distances, $v + \phi_{\text{elect}} - v_{\text{elect}}$ has an algebraic tail originating from $W = -\beta_{ij}(v - v_{\text{elect}})$. Thus, contrarily to the other F bonds, F_R decreases algebraically at large distances as

$$F_R(\mathscr{L}_i,\mathscr{L}_j) \underset{R_{ij}\to\infty}{\sim} e^{W(\mathscr{L}_i,\mathscr{L}_j)} - 1. \tag{5.31}$$

This algebraic behavior starts with a $1/r^3$ tail and its inverse power-law expansion is derived from the decomposition

$$W = \sum_{\gamma \geqslant 3} W_\gamma(\mathscr{L}_i,\mathscr{L}_j), \tag{5.32}$$

where W_γ decreases as $1/R_{ij}^{\gamma}$ when R_{ij} goes to infinity,

$$\begin{aligned}W_\gamma(\mathscr{L}_i,\mathscr{L}_j) \equiv &-\beta_{ij}\int_0^{p_i} d\tau \int_0^{p_j} d\tau' \{\delta([\tau - P(\tau)]\\ &-[\tau' - P(\tau')]) - 1\}\frac{1}{(\gamma-1)!}\\ &\times[\mathbf{X}_i(\tau)\cdot\boldsymbol{\nabla}_i + \mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_j]^{\gamma-1} v_C(R_{ij}).\end{aligned} \tag{5.33}$$

The leading term W_3 reads

$$\begin{aligned}W_3(\mathscr{L}_i,\mathscr{L}_j) \equiv &\beta_{ij}\int_0^{p_i} d\tau \int_0^{p_j} d\tau'\\ &\times\{\delta([\tau - P(\tau)] - [\tau' - P(\tau')]) - 1\}\\ &\times[\mathbf{X}_i(\tau)\cdot\boldsymbol{\nabla}_{\mathbf{R}_{ij}}][\mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_{\mathbf{R}_{ij}}]v_C(R_{ij}).\end{aligned} \tag{5.34}$$

W_γ involves the shapes $\mathbf{X}_i$ and $\mathbf{X}_j$ of $\mathscr{L}_i$ and $\mathscr{L}_j$ simultaneously, because of the property

$$\int_0^{p_i} d\tau \int_0^{p_j} d\tau' \{\delta([\tau - P(\tau)] - [\tau' - P(\tau')]) - 1\} g(\tau) = 0. \tag{5.35}$$

Our procedure is analogous to the resummations of the Coulomb divergencies that are done in the density-expansion scheme of Ref. [25] for a multicomponent plasma where the exchange effects are treated perturbatively. The latter formalism leads to the *exact* calculation of the equation of state up to order $\frac{5}{2}$ in the densities. These calculations are valid in the internal shells of the sun, where the density is low enough and the temperature is sufficiently high for the sizes of the

Brownian bridges to be lower than the interparticle distance (weak degeneracy). In this scheme, some resummed bonds decrease exponentially as either the Debye-Hückel potential (classical collective screening) or its gradient (''diffraction'' effects of quantum dynamics) and the other bonds decrease as $1/r^3$ and involve the description of the quantum two-body bound states. The topological definitions of the resummed bonds are not the same as in the present paper; they are not convenient for the study of the large-distance decay of the correlations because they only exhibit the exponential screening of the charge-charge and charge-dipole interaction, whereas, in fact, every multipole-charge interaction is exponentially screened, as in the classical regime. (Moreover, the exchange effects are not taken into account systematically in this formalism.)

In the classical limit, our formalism leads to the resummation scheme with reexponentiation that Meeron developed for the density expansion of the correlations in ionic solutions [2]. As shown in Sec. III B, when $\hbar$ goes zero, the loops are reduced to points that interact through the potential $e_{\alpha_i} e_{\alpha_j} v_C(r_{ij})$. The screening length tends to the classical Debye length $\kappa_{\mathrm{DH}}^{-1}$. The parameter τ does not appear in the interaction, v_{elect} coincides with v, and $W=-\beta_{ij}(v-v_{\mathrm{elect}})$ vanishes. Thus F^{cm} and F^{mc} do not exist, while $F^{cc}|_{\hbar=0}=-\beta_{ij}\phi_{\mathrm{DH}}(r_{ij})$ and $F_R|_{\hbar=0}=\exp[-\beta_{ij}\phi_{\mathrm{DH}}(r_{ij})]-1+\beta_{ij}\phi_{\mathrm{DH}}(r_{ij})$ coincides with the resummed bars introduced by Meeron. The topological definition of the resummed diagrams is the same as in the present formalism.

D. Decay of the exchange part of the correlations

As shown in Appendix B, a resummation in the loop-fugacity expansion (5.2) of $\rho(\mathscr{L}_a)$ can be performed along a process similar to that in Sec. V B for the loop-density expansion of the Ursell function. The main difference comes from the existence of articulation points in the $\mathbb{G}$ diagrams involved in formula (5.2). This difference has two consequences. First, there may be articulation points also in the prototype diagrams $\mathbb{P}$ obtained after the resummation. Second, convolution rings that are attached to an articulation point and in which all the intermediate points are Coulomb points disappear when all these points are integrated over. In the following, these rings are called ''Coulomb rings.'' Subsequently, the points $\mathscr{P}$ that are left over have a weight $w(\mathscr{P})$ different from the value $z(\mathscr{P})$ that they had in the $\tilde{\mathbb{G}}$ diagrams. The analog of the resummation formula (5.10) reads

$$\rho(\mathscr{L}_a)=w(\mathscr{L}_\alpha)\sum_{\mathbb{P}}\frac{1}{S_{\mathbb{P}}}\int\prod_{m=1}^{M}[d\mathscr{P}_m w(\mathscr{P}_m)]\Big[\prod F_z\Big]_{\mathbb{P}}. \tag{5.36}$$

The sum runs over all the unlabeled topologically different connected diagrams $\mathbb{P}$ with one root point $\mathscr{L}_a$ and in which two points are linked by at most one F_z bond. Since there must be a one-to-one correspondence between $\mathbb{P}$ diagrams and a partition of the $\mathbb{G}$ diagrams into equivalence classes, excluded-convolution rules appear and, at the same time, the weight $w(\mathscr{P}_m)$ of every internal point $\mathscr{P}_m$, depends on the role of P_m in the topology of the diagram $\mathbb{P}$. As shown in Appendix B, the excluded-convolution rules for the $\mathbb{P}$ diagrams are simply expressed by introducing two values for the weight w (except in the case of $\mathscr{L}_a$, which has the same role in all $\mathbb{P}$ diagrams) and are analogous to those of Sec. V B. The formula (5.36) is similar to Eq. (5.3) in Ref. [25]. In order to get finite weights, the auxiliary decomposition of f into $\tilde{f}$ bonds is slightly different from (5.6) and five F_z bonds appear. The F_z bonds decay either exponentially, with an inverse screening length κ_z, or algebraically, as $\exp[W]-1$. The expression of κ_z is the same as that of κ, with $z_{\alpha,p}(\mathbf{X})$ in place of $\rho_{\alpha,p}(\mathbf{X})$, and, in the bosonic case, it is lower than the RPA value. After resummation of the charge-charge Coulomb divergencies, according to (B9), (B11), and (B12), the fugacity of the root point $\mathscr{L}_a$ is replaced by a weight $w(\mathscr{L}_a)$, which is lower than the loop fugacity corresponding to a quantum system where the interaction between particles is $\phi_z(r)=\exp(-\kappa_z r)/r$ instead of $v_C(r)$ and the chemical potential of a particle is equal to μ_α plus the self-energy of a point charge creating a potential ϕ_z,

$$|w(\mathscr{L}_a)|<|z^*_{\alpha_a,p_a}(\mu_{\alpha_a}+\tfrac{1}{2}e^2_{\alpha_a}\kappa_z)|e^{-E_\beta^{\mathrm{int}\phi_z}(\{\mathbf{t}_l^a\})}. \tag{5.37}$$

Eventually, according to (3.16), the weight of the root point $\mathscr{L}_a$ in a $\mathbb{P}$ diagram is bounded by the fugacity of a noninteracting loop with a chemical potential that includes the self-energy of a point particle creating a Debye-Hückel potential $\exp(-\kappa_z r)/r$,

$$|w(\mathscr{L}_a)|<|z^*_{\alpha_a,p_a}(\mu_{\alpha_a}+\tfrac{1}{2}e^2_{\alpha_a}\kappa_z)|e^{-E_\beta^0(\{\mathbf{t}_l^a\})}. \tag{5.38}$$

The integrability of the resummed diagrams in both the loop-fugacity expansion of the loop-density and the loop-density expansion of the loop Ursell function is studied in Appendix C. The two values for the weight of internal points are shown to be bounded by a constant times the Gaussian $\exp[-E_\beta^0(\{\mathbf{t}_l\})]$, according to (3.16) and (B13), and the measure over the Brownian bridges $\{\boldsymbol{\xi}_l\}_{l=1,\dots,p}$ is also Gaussian. Subsequently, the integrals corresponding to the resummed $\mathbb{P}$ diagrams are shown to be conditionally convergent at large distances if the integration over the shapes $\mathbf{X}$ of the internal loops are performed before the integration over the relative positions of the loops. (This kind of procedure also operates in the case of classical two-dimensional plasmas without any resummation [39].) Then, the nonabsolutely integrable part of the asymptotic behavior of the algebraic bond disappears partially because of rotational invariance arguments and the remaining contribution from dipole-dipole-like interactions is proportional to the short-ranged Laplacian of the Coulomb potential. The spurious short-distance singularities due to the auxiliary decomposition of f into $\tilde{f}$ bonds disappear when the diagrams are suitably collected together [40,41]. As argued in Appendix C, the loop density $\rho(\mathscr{P})$ is expected to decay faster than any inverse power law of the extension of the shape $\mathbf{X}$, so that the integrability of the Π diagrams in the loop-density expansion of the Ursell function is readily derived from the connectivity of those diagrams.

Since the average of the loop density over the Brownian bridges has a fast decay [see (C7)] when the distance between two points in the loop becomes very large, the part (4.6) of the particle-particle correlation that comes from exchange effects is bounded, according to (3.19) by

$$
\begin{aligned}
|\rho^{(2)TQ}_{\alpha_a\alpha_a}|_{\text{exch}}(r_{ab})| < \sum_{p_a=2}^{+\infty} p_a \int \left[\prod_{l=1}^{p_a} d\mathbf{t}_l^a\right] \sum_{l^*=2}^{p_a} \\
\times \delta\left(\sum_{l=1}^{l^*-1} \mathbf{t}_l^a - \mathbf{r}_{ab}\right) \delta\left(\sum_{l=l^*}^{p_a} \mathbf{t}_l^a + \mathbf{r}_{ab}\right) \\
\times |z^*_{\alpha_a,p_a}(\mu_{\alpha_a} + \tfrac{1}{2} e^2_{\alpha_a}\kappa_z)| \\
\times \exp(\\
-\frac{1}{2\lambda^2_{\alpha_a}} \sum_{l=1}^{p_a} [\mathbf{t}_l^a]^2) \int \left[\prod_{l=1}^{p} D(\boldsymbol{\xi}_l^a)\right] \\
\times \left|\sum_{\mathbb{P}} I_{\mathbb{P}}(\mathscr{L}_a)\right|. \qquad (5.39)
\end{aligned}
$$

In Appendix C we argue that

$$\exp\left[-(1/2\lambda^2_{\alpha_a})\sum_{l=1}^{p_a} [\mathbf{t}_l^a]^2\right] \int \left[\prod_{l=1}^{p} D(\boldsymbol{\xi}_l^a)\right] \left|\sum_{\mathbb{P}} I_{\mathbb{P}}(\mathscr{L}_a)\right|$$

decays faster than any inverse power law of $\{\mathbf{t}_l^a\}_{l=1,\ldots,p_a}$. At this point, we may only conjecture that the integration over the positions of p_a-2 particles and the summation over the size p_a of the loop $\mathscr{L}_a$ preserves this fast decay.

To exemplify our discussion, we return to the noninteracting case and we mention a few results about the large-distance behavior of the correlation $\rho^{(2)T0Q}_{\alpha\alpha}$ (4.22) in a noninteracting system. For bosons in the noncondensated phase ($\mu_\alpha<0$) and for fermions ($\eta_\alpha=+1$), $n^0_\alpha(\mathbf{k})$ is infinitely differentiable in $\mathbf{k}$ at finite temperature, its inverse Fourier transform decays faster than any inverse power of r_{ab}, and so does the correlation $\rho^{(2)T0Q}_{\alpha\alpha}$. However, the integral over $\mathbf{k}$ in (4.22) can be written as a series of Gaussians with increasing ranges and it is not obvious that the sum of the series eventually has an exponential falloff.

For instance, for bosons at any temperature larger than the Bose-condensation critical value, the analysis in the complex plane of $|\mathbf{k}|$ shows that the leading term in the asymptotic behavior of the correlation $\rho^{(2)T0Q}_{\alpha\alpha}$ (4.22) decays exponentially (without any oscillating factor) over a length scale $[2k^*_\alpha]^{-1}$, where k^*_α is defined in a similar way to the Fermi momentum for fermions $\hbar^2 k^{*2}_\alpha/2m_\alpha=|\mu_\alpha|$. On the other hand, the integral over $\mathbf{k}$ in $\eta_\alpha G^0_\alpha(r_{ab})$ may be written as a series of Gaussians

$$
\begin{aligned}
\int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}_{ab}} \sum_{q=1}^{\infty} (\eta_\alpha e^{\beta\mu_\alpha})^q e^{-q(\lambda^2_\alpha \mathbf{k}^2/2)} \\
= \sum_{q=1}^{\infty} (\eta_\alpha e^{\beta\mu_\alpha})^q \left(\frac{1}{2\pi q\lambda^2_\alpha}\right)^{3/2} e^{-(1/2q\lambda^2_\alpha)r^2_{ab}}. \qquad (5.40)
\end{aligned}
$$

This series is convergent for any given r_{ab} because $\exp(\beta\mu_\alpha)<1$. The range of the Gaussians increases with q and the large-distance behavior of the sum of the series has a leading term that decays only exponentially. Under the assumption that the same mechanism takes place for the upper bound in (5.39), the latter inequality ensures that $\rho^{(2)TQ}_{\alpha_a\alpha_a}|_{\text{exch}}(r_{ab})$ is a rapidly decreasing function of r_{ab}.

As a final remark, we stress that this fast decay might disappear at strictly zero temperature. For instance, in a ideal gas of fermions at zero temperature, $n^0_\alpha(\mathbf{k})$ is a (nonanalytic) step function and the correlation function (4.22) decays algebraically with an oscillating factor that has a phase $2k_{F,\alpha}r_{ab}$, where $k_{F,\alpha}$ is the Fermi momentum $\hbar^2 k^2_{F,\alpha}/2m_\alpha=\mu_\alpha$. However, at any nonzero temperature, however small it is, the large-distance behavior of the particle-particle correlation is given by the poles of $n^0_\alpha(\mathbf{k})$ (in the complex plane of the variable $|\mathbf{k}|$) that have the smallest imaginary part and the oscillations are damped by an exponential factor over a distance $\beta\hbar^2 k_{F,\alpha}/2\pi m_\alpha$. This difference in the analyticity properties of the occupation number when the temperature is increased from zero to a finite value also generates the Friedel oscillations in the large-distance behavior of the one-component plasma density around a test charge at zero temperature in the RPA model, and its damping, as soon as the temperature is nonzero (see p. 179 in Ref. [36] and references cited therein).

VI. COMPARISON WITH THE STANDARD PERTURBATION FORMALISM

A. General structures of the diagrams

At finite temperature, as recalled in Sec. IV C, the quantum static structure factor $S^Q_{\alpha\alpha'}(\mathbf{r})\equiv\rho^{(2)TQ}_{\alpha\alpha'}(\mathbf{r})+\delta_{\alpha,\alpha'}\rho^Q_\alpha\delta(\mathbf{r})$ can be calculated as the opposite of the time-ordered density-density correlation at equal times [36] and the induced charge density in the presence of an infinitesimal external charge can be related through the linear response theory (4.34) to the integral of the time-ordered charge-charge correlation function $C_T(\mathbf{r},s)$ over the time s.

In the standard many-body perturbation theory, with the normalizations of Ref. [21], the time-ordered density-density correlation in imaginary time is equal to the opposite of the total "polarization" $\text{Л}(\mathbf{r},s)$ [36]. The usual notation Л for the polarization is not to be confused with the notation Π for the prototype diagrams of Sec. V. The total polarization is the sum of all the connected Feynman-like graphs in which the points $(\mathbf{r},s)$ and $(\mathbf{0},0)$ are linked by loops of free propagators $G^0_\alpha(\mathbf{r},s_1-s_2)$ joined by interaction lines $U^0_{\alpha\alpha'}(\mathbf{r},s'_1-s'_2)=-\beta e_\alpha e_{\alpha'} v_C(\mathbf{r})\delta(s'_1-s'_2)$, according to some prescribed rules [42]

$$G^0_\alpha(\mathbf{r},s)=\int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} e^{s(\beta\mu_\alpha-\lambda^2_\alpha\mathbf{k}^2/2)}[n^0_\alpha-\theta(s)], \qquad (6.1)$$

where $\theta(s)$ is the Heaviside function. The graphs are conveniently calculated by Fourier transform over both the position and time variables. For brevity, we use the notation $\text{Л}(\mathbf{k},n)\equiv\text{Л}(\mathbf{k},\nu_n=2\pi n)$ in the following. The total polarization Л can be calculated in terms of a more basic object of the perturbation theory, the "proper" polarization Л*, which is the sum of all the polarization graphs that are not split into two pieces when one interaction line is suppressed. According to the topological definition of Л*, Л is the sum of all the chains built with proper polarization graphs linked by interaction lines,

$$\Pi_{\alpha\alpha'}(\mathbf{k},n)=\frac{\Pi^{*}_{\alpha\alpha'}(\mathbf{k},n)}{1-\beta v_C(\mathbf{k})\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}\Pi^{*}_{\alpha\alpha'}(\mathbf{k},n)}. \tag{6.2}$$

The definition of the proper polarization is related to that of the ‘‘effective’’ potential $U^{\text{eff}}_{\alpha\alpha'}(\mathbf{k},n)$: $U^{\text{eff}}_{\alpha\alpha'}(\mathbf{k},n)$ is the sum of all the chain Feynman diagrams built with interaction lines linked by ‘‘proper polarization’’ graphs,

$$U^{\text{eff}}_{\alpha\alpha'}(\mathbf{k},n)=\frac{\beta e_\alpha e_{\alpha'} v_C(\mathbf{k})}{1-\beta v_C(k)\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}\Pi^{*}_{\alpha\alpha'}(\mathbf{k},n)}. \tag{6.3}$$

Equation (6.2) can be rewritten as

$$\Pi_{\alpha\alpha'}(\mathbf{k},n)=\Pi^{*}_{\alpha\alpha'}(\mathbf{k},n)+\sum_{\alpha_1,\alpha'_1}\Pi^{*}_{\alpha\alpha_1}(\mathbf{k},n)U^{\text{eff}}_{\alpha_1\alpha'_1}(\mathbf{k},n)\Pi^{*}_{\alpha'_1\alpha'}(\mathbf{k},n). \tag{6.4}$$

The quantum static structure factor, which is given by a relation similar to (4.35), is related to the polarization graphs via

$$S^{Q}_{\alpha\alpha'}(\mathbf{k})=\rho^{(2)TQ}_{\alpha\alpha'}(\mathbf{k})+\delta_{\alpha,\alpha'}\rho^{Q}_{\alpha}=-\sum_{n=-\infty}^{+\infty}\Pi_{\alpha\alpha'}(\mathbf{k},n). \tag{6.5}$$

According to (4.34), the static induced charge density in the presence of an external charge distribution $\delta q(\mathbf{r})$ is related to the zero-frequency component of the polarization

$$\begin{aligned}\frac{\sum_{\alpha} e_\alpha\rho^{\text{ind}\,Q}_{\alpha}(\mathbf{k})}{\delta q(\mathbf{k})}&=\beta v_C(\mathbf{k})\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}\Pi_{\alpha\alpha'}(\mathbf{k},n=0)\\&=\frac{4\pi\beta\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}\Pi^{*}_{\alpha\alpha'}(\mathbf{k},n=0)}{\mathbf{k}^2-4\pi\beta\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}\Pi^{*}_{\alpha\alpha'}(\mathbf{k},n=0)}.\end{aligned} \tag{6.6}$$

The latter equation ensures that, if the small-$\mathbf{k}$ expansion of $\Pi^{*}(\mathbf{k},n=0)$ starts with a power of $|\mathbf{k}|$ lower than 2, then $\Sigma_{\alpha}e_\alpha\rho^{\text{ind}}_{\alpha}(\mathbf{k}=\mathbf{0})=-\int d\mathbf{r}\;\delta q(\mathbf{r})$. The total potential V^{tot} in the bulk in the presence of the external charge $\delta q(\mathbf{r})$ is related to the induced charge density by the Poisson equation

$$\Delta V^{\text{tot}}(\mathbf{r})=-4\pi\left[\delta q(\mathbf{r})+\sum_{\alpha} e_\alpha\rho^{\text{ind}\,Q}_{\alpha}(\mathbf{r})\right]. \tag{6.7}$$

Therefore, according to the linear response equation (6.6) and (6.3), $V^{\text{tot}}(\mathbf{k})$ is proportional to the zero-frequency component of the effective potential

$$\begin{aligned}\frac{V^{\text{tot}}(\mathbf{k})}{\delta q(\mathbf{k})}&=\frac{1}{\beta e_\alpha e_{\alpha'}}U^{\text{eff}}_{\alpha\alpha'}(\mathbf{k},n=0)\\&=\frac{4\pi}{\mathbf{k}^2-4\pi\beta\sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}\Pi^{*}_{\alpha\alpha'}(\mathbf{k},n=0)}.\end{aligned} \tag{6.8}$$

The correlation function and the induced charge density can also be written in terms of both the effective potential and the proper polarization by using the relation (6.4). For instance, in space and time representation, the relation (6.4) allows one to rewrite (6.5) as

$$\begin{aligned}\rho^{(2)TQ}_{\alpha\alpha'}(\mathbf{r})+\delta_{\alpha,\alpha'}\rho_\alpha\delta(\mathbf{r})=&-\Pi^{*}_{\alpha\alpha'}(\mathbf{r},s=0)-\int d\mathbf{r}_1\int d\mathbf{r}'_1\int_0^1 ds_1\int_0^1 ds'_1\sum_{\alpha_1}\sum_{\alpha'_1}\Pi^{*}_{\alpha\alpha_1}(\mathbf{r}_1,s_1)\\&\times U^{\text{eff}}_{\alpha_1\alpha'_1}(\mathbf{r}+\mathbf{r}'_1-\mathbf{r}_1,s'_1-s_1)\Pi^{*}_{\alpha'_1\alpha'}(-\mathbf{r}'_1,-s'_1).\end{aligned} \tag{6.9}$$

In the noninteracting case, (6.9) becomes $\rho^{(2)T0Q}_{\alpha\alpha}(\mathbf{r})+\rho^{0}_{\alpha}\delta(\mathbf{r})=-\Pi^{0}_{\alpha}(\mathbf{r},s=0)$ with $\Pi^{0}_{\alpha}(\mathbf{r},s)=-\eta_\alpha(2S_\alpha+1)G^{0}_{\alpha}(\mathbf{r},s)G^{0}_{\alpha}(-\mathbf{r},-s)$ and, according to (6.1), the result (4.22) is retrieved. From a formal point of view, the analog of the Dyson-like equation (6.9) is Eq. (4.5), written as

$$\begin{aligned}\rho^{(2)TQ}_{\alpha\alpha'}(\mathbf{r})+\delta_{\alpha,\alpha'}\rho_\alpha\delta(\mathbf{r})=&\,\delta_{\alpha,\alpha'}\left\{\rho_\alpha\delta(\mathbf{r})+\sum_{p=2}^{\infty}p\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})\sum_{l^{*}=2}^{p}\delta\left(\mathbf{r}-\sum_{l=1}^{l^{*}-1}\mathbf{t}_l\right)\right\}\\&+\sum_{p=1}^{\infty}p\sum_{p'=1}^{\infty}p'\int D(\mathbf{X})\int D(\mathbf{X}')\rho_{\alpha,p}(\mathbf{X})h_{\alpha,p;\alpha',p'}(\mathbf{r},\mathbf{X},\mathbf{X}')\rho_{\alpha',p'}(\mathbf{X}'),\end{aligned} \tag{6.10}$$

where $h_{\alpha,p;\alpha',p'}(\mathbf{k},\mathbf{X},\mathbf{X}')$ is the Ursell function between two loops. Similarly, the linear response equation (6.6), in time representation and with Π written in terms of Π^* and U^{eff} by using (6.4), is the analog of the expression (4.27) for the induced charge density in the loop formalism, with $\rho_{\alpha,p}(\mathbf{X})h_{\alpha,p;\alpha',p'}(\mathbf{r},\mathbf{X},\mathbf{X}')\rho_{\alpha',p'}(\mathbf{X}')$ in place of $\rho^{(2)T}_{\alpha,p;\alpha',p'}(\mathbf{r},\mathbf{X},\mathbf{X}')$. However, the formal similarity is not so complete as to allow a one-to-one correspondence between objects of both formalisms.

The perturbation Mayer formalism developed for the loops in Sec. V differs from the standard many-body perturbation theory in two main respects. First, in the standard theory, the reference system is the noninteracting gas, whereas, in the present formalism, the interaction between the charges that are exchanged is taken into account nonperturbatively from the start. Second, in the standard many-body perturbation theory, the basic objects associated with the particles are free-propagator loops and the potential between the latter ones involves only a *discrete* number of points in the propagators, whereas, in the potential between exchange loops, *every* line element of both curves interacts with at least another one. This second difference is crucial and implies that we could not get a one-to-one correspondence between the two diagrammatics, even if we introduced another decomposition of the Mayer bond into an infinite sum of bonds,

$$f(\mathscr{L},\mathscr{L}')=\sum_{N=1}^{\infty}\frac{[-\beta v(\mathscr{L},\mathscr{L}')]^N}{N!},\tag{6.11}$$

and if we expanded the weight $\rho(\mathscr{L})$ about its value $\rho^0(\mathscr{L})=z^0(\mathscr{L})$ for the noninteracting system. However, in the Maxwell-Boltzmann statistics approximation [where only exchange loops $\mathscr{L}^{\alpha,1}=\mathscr{E}^{\alpha}$ with $p=1$ survive; see (3.17)] only one Brownian bridge is associated with each point of a Mayer diagram and a correspondence can be sketched between some Fourier transform over the Brownian bridges and the loop $\Pi^{0,\text{MB}}_{\alpha}(\mathbf{k},s)$ made of two free propogators in the Maxwell-Boltzmann limit. $\Pi^{0,\text{MB}}_{\alpha}(\mathbf{k},s)$, which is the value of Π^0_{α} in the regime $\lambda^2_{\alpha}\mathbf{k}^2/2\gg\beta\mu_{\alpha}$, is obtained by replacing $n^0_{\alpha}(\mathbf{k})$ by $e^{\beta\mu_{\alpha}}e^{-\lambda^2_{\alpha}\mathbf{k}^2/2}$ and $1-n^0_{\alpha}(\mathbf{k})$ by 1. For instance, in Ref. [21], the case of the OCP is dealt with as follows. The decomposition $f(\mathscr{E}_i,\mathscr{E}_j)=\Sigma_{n=1}^{\infty}[-\beta v(\mathscr{E}_i,\mathscr{E}_j)]^n/n!$ of the Mayer bond between two filaments $\mathscr{E}_i$ and $\mathscr{E}_j$ is used in the Mayer graphs of the filament-fugacity expansions. In those graphs, each point has a weight $z^*_{\alpha,1}$, which is equal to the density $\rho^{0,\text{MB}}_{\alpha}(\beta,\mu_{\alpha})$ of a noninteracting gas with Maxwell-Boltzmann statistics. This weight $z(\mathscr{E})=z^*_{\alpha,1}$ is a constant independent from $\mathscr{E}$ [contrary to the density of closed filaments $\rho^{\text{MB}}(\boldsymbol{\xi})$] and, though all the $\tilde{f}$ bonds in this decomposition explicitly depend on the shapes of the two filaments, the integration over the shapes of the filaments in the convolution chains of bonds $-\beta e^2 v(\mathscr{E},\mathscr{E}')$ can be done explicitly. Moreover, since only one Brownian bridge is associated with each point of a graph, the Fourier transform of the convolution $\int d\mathbf{R}_i\int D(\boldsymbol{\xi}_i)v(\mathscr{E}_{i-1},\mathscr{E}_i)v(\mathscr{E}_i,\mathscr{E}_{i+1})$ over the position variable $\mathbf{R}_{i+1}-\mathbf{R}_{i-1}$ is proportional to $[v_C(\mathbf{k})]^2\int D(\boldsymbol{\xi}_i)\exp\{i\lambda_{\alpha}\mathbf{k}\cdot[\boldsymbol{\xi}_i(s_i')-\boldsymbol{\xi}_i(s_i)]\}$, which is a function of the difference $s_i'-s_i$. More precisely, $z'_{\alpha,1}\int D(\boldsymbol{\xi})\exp\{i\lambda_{\alpha}\mathbf{k}\cdot[\boldsymbol{\xi}(s')-\boldsymbol{\xi}(s)]\}=\rho^{0,\text{MB}}_{\alpha}\exp[-(\lambda^2_{\alpha}\mathbf{k}^2/2)|s-s'|(1-|s-s'])]=\eta_{\alpha}\Pi^{0,\text{MB}}_{\alpha}(\mathbf{k},s-s')$.

B. Coulombic-divergency resummations and chain potentials

In order to get finite graphs in the various diagrammatics, the Coulomb divergencies are resummed, at least partially, by introducing a chain potential. In the classical case, the chain potential [15] is the sum of all the convolutions of bonds $-\beta e_{\alpha}e_{\alpha'}v_C$ with a weight ρ_{α} for each intermediate point. The classical chain potential is equal to $-\beta e_{\alpha}e_{\alpha'}$ times the Debye-Hückel potential $\phi_{\text{DH}}(r)=\exp[-\kappa_{\text{DH}}r]/r$. It appears in the Mayer-Abe formalism as well as in the exponentiated version developed by Meeron. In both cases, two points in a diagram can be linked by at most one bond and there can be no convolution of bonds $-\beta e_{\alpha}e_{\alpha'}\phi_{\text{DH}}$. The standard perturbation many-body formalism is analogous to Mayer formalism in the sense that the auxiliary bonds in the latter are $(-\beta e_{\alpha}e_{\alpha'}v_C)^N/N!$, with $N=1,\ldots,\infty$, and the resummed bonds are $(-\beta e_{\alpha}e_{\alpha'}\phi_{\text{DH}})^N/N!$. (The bonds with $N\geqslant 2$ can be viewed as products of the more elementary bond $-\beta e_{\alpha}e_{\alpha'}\phi_{\text{DH}}$ arising from the resummation of single chains because these bonds involve the proper symmetry factor $N!$.) On the contrary, the loop formalism is to be compared rather with Meeron resummation scheme, where there are only two kinds of bonds: the auxiliary bonds are $-\beta e_{\alpha}e_{\alpha'}v_C$ and $f+\beta e_{\alpha}e_{\alpha'}v_C$ and the resummed bonds are $-\beta e_{\alpha}e_{\alpha'}\phi_{\text{DH}}$ and $\exp(-\beta e_{\alpha}e_{\alpha'}\phi_{\text{DH}})-1+\beta e_{\alpha}e_{\alpha'}\phi_{\text{DH}}$.

In the standard many-body theory, the same kind of resummation process as in the classical Mayer formalism can be introduced, in order to deal with the nonintegrability of the Coulomb potential. The chain potential is the sum of all the chains of interaction lines linked by free proper polarizations $\Pi^0_{\alpha\alpha'}=\delta_{\alpha,\alpha'}\Pi^0_{\alpha}$. It proves to be equal to the so-called RPA potential (which can be derived along other lines [30])

$$-\beta e_{\alpha}e_{\alpha'}\phi_{\text{RPA}}(\mathbf{k},n)=\frac{-\beta e_{\alpha}e_{\alpha'}v_C(\mathbf{k})}{1-\beta v_C(\mathbf{k})\sum_{\alpha}e^2_{\alpha}\Pi^0_{\alpha}(\mathbf{k},n)}.\tag{6.12}$$

At large distances, the zero-frequency component of ϕ_{RPA} decays faster than any inverse power-law with a leading term

$$\phi_{\text{RPA}}(\mathbf{r},n=0)\underset{r\to\infty}{\sim}Z_{\text{RPA}}\frac{e^{-(Z_{\text{RPA}}\kappa_{\text{RPA}})r}}{r},\tag{6.13}$$

where $\kappa^2_{\text{RPA}}=-4\pi\beta\Sigma_{\alpha}e^2_{\alpha}\Pi^0_{\alpha\alpha}(\mathbf{k}=\mathbf{0},n=0)=4\pi\beta\Sigma_{\alpha}e^2_{\alpha}\partial\rho^{0Q}(\beta,\mu_{\alpha})/\partial(\beta\mu_{\alpha})$. Z_{RPA} is a renormalization factor of the charge [21]

$$Z_{\text{RPA}}=\left[-\frac{\pi\hbar^2}{3}\sum_{\alpha}\frac{e^2_{\alpha}}{m_{\alpha}}\frac{\partial^2\rho^0_{\alpha}}{(\partial\mu_{\alpha})^2}\right]^{-1/2}.\tag{6.14}$$

Though the zero-frequency component of the RPA chain potential $\phi_{\text{RPA}}(\mathbf{r},s)$ decays faster than any inverse power law, the nonzero-frequency components are purely Coulombic [21], $\phi_{\text{RPA}}(\mathbf{k},n\neq 0)=h(n)v_C(\mathbf{k})$ and

$$\sum_{n\neq 0}e^{-i2\pi n(s_1-s_2)}\phi_{\text{RPA}}(\mathbf{r},n)=\frac{1}{r}h(s_1-s_2),\tag{6.15}$$

where $h(s_1-s_2)\equiv\Sigma_{n\neq 0}e^{-i2\pi n(s_1-s_2)}h(n)$ has the property (5.35).

As pointed out in Sec. VI A, in the present paper we use a decomposition of f into auxiliary bonds that is different from (6.11) and, in the loop-fugacity [loop-density] expansions, the weight of a loop is $z_{\alpha,p}(\mathbf{X})[\rho_{\alpha,p}(\mathbf{X})]$ instead of a series of graphs where every point would have a weight equal to $z^0_{\alpha,p}(\mathbf{X})=\rho^0_{\alpha,p}(\mathbf{X})$. The difference between the formulas for κ^2_{RPA} (6.13) and

$$\kappa^2=4\pi\beta\sum_\alpha e^2_\alpha\frac{\partial\rho^Q_\alpha}{\partial(\beta\mu_\alpha)}\bigg|_\beta$$

$$-4\pi\beta\sum_\alpha e^2_\alpha\int d\mathbf{r}\;\rho^{(2)TQ}|_{\text{nonexch}}(\mathbf{r})$$

[see (4.9) and (5.14)] reflects the difference between $\rho^0_{\alpha,p}(\mathbf{X})$ and $\rho_{\alpha,p}(\mathbf{X})$. We notice that, if we consider a system of noninteracting particles with various species α and associated chemical potentials μ^0_α such that its loop density $\rho^0_{\alpha,p}(\mathbf{X})$ satisfies the relation

$$\sum_{p=1}p^2\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})=\sum_{p=1}p^2\int D(\mathbf{X})\rho^0_{\alpha,p}(\mathbf{X})$$

$$=\partial\rho^0_\alpha/\partial(\beta\mu^0_\alpha),$$

then $\kappa^2=4\pi\beta\Sigma_\alpha e^2_\alpha\partial\rho^0_\alpha/\partial(\beta\mu^0_\alpha)$, but this latter relation is useless because the density ρ^0_α cannot be related to ρ_α in a simple way. Indeed, an ideal gas cannot be found such that $\int D_p(\mathbf{X})\rho^0_{\alpha,p}(\mathbf{X})=g^0_\alpha(\beta,\mu^0_\alpha,p)$ might be equal to $\int D_p(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})=g_\alpha(\beta,\{\mu_{\alpha'}\}_{\alpha'=1,\ldots,n_s},p)$ for every p.

Nevertheless, a ready comparison can be made between the decay of the RPA potential in the standard theory and the falloff of the sum of the four elementary bonds in the ''linearized'' version of the loop formalism. In this version, the f bond is decomposed according to (6.11) and (5.8) and no product of $\tilde{f}'$ bond chains is resummed, so that no sum of single chains is exponentiated. As mentioned at the beginning of this subsection, in this resummation process, the excluded-convolution rules still hold and two points can be linked by at most one multiple bond. The four elementary bonds are merely the sum of the four kinds of single chains, namely, F^{cc}, F^{mc}, F^{cm}, and F^{mm}, according to the diagrammatic definitions of Sec. V B. [F^{mm} does coincide with the linearized value of F_R (5.22) with respect to the argument in the exponential.] According to (5.28), the sum of the four bonds is equal to

$$-\beta e_\alpha e_{\alpha'}\phi_{\text{linearized chain}}(\mathscr{L},\mathscr{L}')=-\beta e_\alpha e_{\alpha'}\phi_{\text{elect}}(\mathscr{L},\mathscr{L}')$$

$$+W(\mathscr{L},\mathscr{L}'). \qquad (6.16)$$

$\phi_{\text{elect}}(\mathbf{r},\mathbf{X},\mathbf{X}')$ decays faster than any inverse power law of the distance, whereas $W(\mathbf{r},\mathbf{X},\mathbf{X}')$ has an algebraic falloff. Henceforth, $\phi_{\text{elect}}(\mathbf{r},\mathbf{X},\mathbf{X}')$ is to be compared with $\phi_{\text{RPA}}(\mathbf{r},n=0)=\int_0^1 ds\;\phi_{\text{RPA}}(\mathbf{r},s)$, while $W(\mathbf{r},\mathbf{X},\mathbf{X}')$ is analogous to $\Sigma_{n\neq 0}\exp(-i2\pi ns)\phi_{\text{RPA}}(\mathbf{r},n)$.

As a conclusion, the possibility of algebraic tails in the quantum correlations already appears in chain potentials. In the RPA theory, they lie in the nonzero-frequency components of ϕ_{RPA} and, in the loop formalism, they originate from $W(\mathbf{r},\mathbf{X},\mathbf{X}')$, where the function $\delta(\tau-P(\tau)-[\tau'-P(\tau')])-1$ has the same property as $h(s,s')$ [see (6.15)]. The analogy between the two functions can be displayed more precisely by considering the Maxwell-Boltzmann limit of the RPA theory.[21] The Maxwell-Boltzmann limit of ϕ_{RPA} can be obtained directly by considering the Mayer diagrams of the fugacity expansion for the equivalent gas of closed filaments $\mathscr{E}$ (see Sec. VI A) and by performing chain summations analogous to those performed in the quantum RPA theory as well as in the fugacity-expansion diagrammatic version of the classical Debye-Hückel model. The corresponding chain potential reads

$$\phi^{\text{eff}}(\mathbf{r}_{ij},\boldsymbol{\xi}_i,\boldsymbol{\xi}_j)=\int_0^1 ds\int_0^1 ds'$$

$$\times\phi^{\text{MB}}_{\text{RPA}}(\mathbf{r}_{ij}+\lambda_{\alpha_i}\boldsymbol{\xi}_i-\lambda_{\alpha_j}\boldsymbol{\xi}_j,s-s'),$$

$$(6.17)$$

where $\phi^{\text{MB}}_{\text{RPA}}(\mathbf{r},s)$ proves to be the Maxwell-Boltzmann limit of $\phi_{\text{RPA}}(\mathbf{r},s)$ [see (6.12)]. $\phi^{\text{eff}}(\mathbf{r}_{ij},\boldsymbol{\xi}_i,\boldsymbol{\xi}_j)$ can be split into a short-ranged part that decreases exponentially (it is the Maxwell-Boltzmann approximation of the contribution from the zero frequency of the RPA potential) and a long-ranged part equal to $\int_0^1 ds\int_0^1 ds' h_{\text{MB}}(s,s')v_C(r_{ij}+\lambda_{\alpha_i}\boldsymbol{\xi}_i-\lambda_{\alpha_j}\boldsymbol{\xi}_j)$ [see (6.15)]. In the limit $\hbar\rightarrow 0$, $h_{\text{MB}}(s,s')$ becomes equal to $\delta(s-s')-1$ and the long-ranged part of ϕ^{eff} becomes equal to $W(\mathscr{L}^{\alpha_i,1},\mathscr{L}^{\alpha_j,1})$. Thus the dominant asymptotic behavior of ϕ^{eff} at the first order in $\hbar$ is equal to that of W for loops with a size $p=1$, namely,

$$W_3(\mathscr{L}^{\alpha_i,1},\mathscr{L}^{\alpha_j,1})=\beta_{ij}\int_0^1 ds\int_0^1 ds'[\delta(s-s')-1][\lambda_{\alpha_i}\boldsymbol{\xi}_j(s)$$

$$\cdot\boldsymbol{\nabla}_{\mathbf{r}_{ij}}]$$

$$\times[\lambda_{\alpha_j}\boldsymbol{\xi}_j(s')\cdot\nabla_{\mathbf{r}_{ij}}]v_C(r_{ij}). \qquad (6.18)$$

C. Inadequacy of the standard perturbation theory

The existence of algebraic tails in the particle-particle correlation can be investigated through the standard perturbation theory only in the very special case of the OCP [21] because the latter obeys two exact specific sum rules linked to the fact that the density of charge in the OCP is proportional to the density of particles. Indeed, it can be shown exactly, by using the equation of mechanical balance for every volumic element [30,37], that

$$\frac{e\rho^{\text{ind}}_{\text{OCP}}(\mathbf{k})}{\delta q}=-1+\frac{\mathbf{k}^2}{4\pi e^2\rho^2\chi_T^2}+o(|\mathbf{k}|^2). \qquad (6.19)$$

χ_T is the isothermal compressibility derived from the thermal pressure [see (2.11)]. On the other hand, the system satisfies the exact sum rule (4.43). Under the assumption that every frequency component $\text{Л}^*(\mathbf{k},n)$ of the exact *proper* polarization $\text{Л}^*(\mathbf{k},s)$ is invariant under rotations of $\mathbf{k}$, it can be shown that, if these frequency components have algebraic decays, so do the corresponding components of the exact *total* polarization $Л(\mathbf{r},s)$: $Л(\mathbf{r},n\neq 0)$ has the same algebraic decay as $Л^*(\mathbf{r},n\neq 0)$ and if $Л^*(\mathbf{r},n=0)$ decays as $1/r^\gamma$, $Л(\mathbf{r},n=0)$ has a $1/r^{\gamma+4}$ falloff.

The argument can be summed up as follows. According to the sum rules (4.43) and (6.5), the small-**k** expansion of $Л(\mathbf{k},n\neq 0)$ starts as $\mathbf{k}^2$ and this term of order $|\mathbf{k}|^2$ is different from $-1/[\beta e^2 v_C(\mathbf{k})]$, which is the corresponding term for the opposite of the classical structure factor, according to the Stillinger-Lovett sum rule (4.42). Subsequently, according to (6.2), the same is true for $Л^*(\mathbf{k},n\neq 0)$, so that the order in $|\mathbf{k}|$ of the first nonanalytic term is the same in the small-**k** expansion of $Л^*(\mathbf{k},n\neq 0)$ and in that of $Л(\mathbf{k},n\neq 0)$: $Л^*(\mathbf{k},n\neq 0)$ and $Л(\mathbf{k},n\neq 0)$ have the same algebraic falloff (see the general discussion of Sec. II C in paper II). $U^{\mathrm{eff}}(\mathbf{r},n\neq 0)$ has a $1/r$ tail, but it does not rule the asymptotic behavior of $Л(\mathbf{k},n\neq 0)$, by virtue of the harmonicity of the Coulomb potential, as displayed in paper II. On the other hand, the sum rule (4.43) implies that the first term in the small-**k** behavior of $Л(\mathbf{k},n=0)$ coincides with the classical value $-1/[\beta e^2 v_C(\mathbf{k})]$ and the comparison of the compressibility sum rule (6.19) with the relation (6.6) implies that the next term is analytic and of order $|\mathbf{k}|^4$. These two properties imply that $Л^*(\mathbf{k},n=0)\neq 0$. As a result, if $Л^*(\mathbf{r},n=0)$ decays as $1/r^\gamma$, $U^{\mathrm{eff}}(\mathbf{r},n=0)$ [see (6.3)] also behaves as $1/r^\gamma$, but $Л(\mathbf{k},n=0)$ falls off only as $1/r^{\gamma+4}$, according to the structure of the fraction in (6.2).

In the case of multicomponent plasmas, on one hand, the screening rule (4.41) does not allow one to get any information about the second moment of $Л_{\alpha\alpha'}(\mathbf{r},n)$ and, on the other hand, an analog of the compressibility sum rule has not been found: the first terms in the small-**k** expansion of $Л_{\alpha\alpha'}(\mathbf{k},n=0)$ are not known exactly. Subsequently, nothing is known about the first terms in the small-**k** behavior of the exact $Л^*_{\alpha\alpha'}(\mathbf{k},n)$ and, even if the proper polarization $Л^*$ of a multicomponent plasma decays as $1/r^\gamma$, nothing can be inferred about the large-distance behavior of $U^{\mathrm{eff}}_{\alpha\alpha'}(\mathbf{r},n)$ or $Л_{\alpha\alpha'}(\mathbf{r},n)$. In fact, as shown in paper II, the particle-particle correlation decays as $1/r^6$ and the small-**k** expansion of $Л_{\alpha\alpha'}(\mathbf{k},n=0)$ involves a nonanalytic term only at the order $|\mathbf{k}|^3$ [whereas, in the case of the OCP, $Л_{\alpha\alpha'}(\mathbf{k},n=0)$ is analytic up to the order $|\mathbf{k}|^4$ included]. Then, according to a mere dimensional analysis of (6.6), the small-**k** expansion of $\Sigma_\alpha e_\alpha \rho_\alpha^{\mathrm{ind}}(\mathbf{k})$ might involve a nonanalytic term of order $|\mathbf{k}|$. However, this nonanalytic term of order $|\mathbf{k}|$, which includes a summation over the species α and α', has a zero coefficient because, as discussed in paper II, the induced charge density decreases only as $1/r^8$ and not as $1/r^4$. The latter result is also obtained from the loop formalism, which allows one to exhibit the classical macroscopic screening in a nicely tractable way. Eventually, the point of view of the Feynman-Kac representation proves to be more efficient than the standard perturbation many-body theory to study the large-distance behavior of correlations in multicomponent plasmas.

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APPENDIX A

In this appendix we show how, for a Hamiltonian independent from the spin, a notion of exchange loops emerges from the fact that the permutations can be collected into classes where all the permutations have the same irreducible cyclic structure. For our purpose, it is convenient to write Ξ in position representation with the following particular choice $\{\|\{\mathbf{r}_i,S^z_{\alpha_i}\}\rangle\}_{i=1,\ldots,\Sigma_\alpha N_\alpha}$ for the basis of quantum states, which must be symmetric (antisymmetric) under the permutations of particles of the same bosonic (fermionic) species. The $\|\{\mathbf{r}_i,S^z_{\alpha_i}\}\rangle$ are chosen to be properly symmetrized tensorial products $\otimes_i$ of the individual particle states $|\mathbf{r}_i,S^z_{\alpha_i}(i)\rangle$,

$$\|\{\mathbf{r}_i,S^z_{\alpha_i}\}\rangle \equiv \sum_{\{\pi_\alpha\}_{\alpha=1,\ldots,n_s}} \prod_\alpha \frac{\epsilon(\pi_\alpha)}{N_\alpha!} \bigotimes_i |\mathbf{r}_{\pi(i)},S^z_{\alpha_i}(\pi(i))\rangle, \tag{A1}$$

where π is the composition of n_s permutations π_α, each of which acts only on particles of species α. For each α, the sum over π_α runs over the whole set of permutations between N_α elements. $\epsilon(\pi_\alpha)=1$ if the particles of species α are bosons, whereas $\epsilon(\pi_\alpha)$ denotes the signature of π_α in the case of fermions. [We notice that, in the definition (A1), the permutations π_α act simultaneously on the position and spinorial variables of each particle, so that the $\|\{\mathbf{r}_i,S^z_{\alpha_i}(i)\}\rangle$'s are not eigenstates of the total spin operator, in general.] In this basis, the infinite-volume limit of (2.3) reads

$$\Xi = \sum_{\{N_\alpha\}_{\alpha=1,\ldots,n_s}} \sum_{\{\pi_\alpha\}_{\alpha=1,\ldots,n_s}} \sum_{\{\pi'_\alpha\}_{\alpha=1,\ldots,n_s}} \prod_\alpha \frac{\epsilon(\pi_\alpha)\epsilon(\pi'_\alpha)e^{\beta\mu_\alpha N_\alpha}}{[N_\alpha!]^2} \int \prod_i d\mathbf{r}_i \sum_{\{S^z_{\alpha_i}(i)\}_{i=1,\ldots,\Sigma_\alpha N_\alpha}}$$
$$\times \left[\bigotimes_i \langle r_{\pi'(i)},S^z_{\alpha_i}(\pi'(i))|\right] e^{-\beta\hat{H}_{\{N_\alpha\}}} \left[\bigotimes_i |\mathbf{r}_{\pi(i)},S^z_{\alpha_i}(\pi(i))\rangle\right], \tag{A2}$$

where $S^z_{\alpha_i}(i)$ can take the values $-S_{\alpha_i},-S_{\alpha_i}+1,\ldots,S_{\alpha_i}$ and $\mathbf{r}_i$ is integrated over an infinite volume. For conciseness, we use the convention that, if $\Sigma_\alpha N_\alpha=0$, there is no integration over positions or summation over spin states and the corresponding contribution is 1. We relabel the index i by a composition of permutations π_α^{-1} and note that $\epsilon(\pi_\alpha)=\epsilon(\pi_\alpha^{-1})$ and $\epsilon(\pi'_\alpha)\epsilon(\pi_\alpha^{-1})=\epsilon(\pi'_\alpha\circ\pi_\alpha^{-1})$ (where $\circ$ denotes the composition of permutations). Then, since the number of permutations π_α between N_α objects is equal to $N_\alpha!$, we can eliminate the sum over the π'_α in (A.2) and write Ξ as

$$\Xi = \sum_{\{N_\alpha\}_{\alpha=1,\ldots,n_s}} \sum_{\{\pi_\alpha\}_{\alpha=1,\ldots,n_s}} \prod_\alpha \frac{\epsilon(\pi_\alpha) e^{\beta\mu_\alpha N_\alpha}}{N_\alpha!} \times \left\{ \int \prod_i d\mathbf{r}_i \sum_{\{S^z_{\alpha_i}(i)\}_{i=1,\ldots,\Sigma_\alpha N_\alpha}} \left[\bigotimes_i \langle \mathbf{r}_{\pi(i)}, S^z_{\alpha_i}(\pi(i)) | \right] e^{-\beta \hat{H}_{\{N_\alpha\}}} \left[\bigotimes_i | \mathbf{r}_i, S^z_{\alpha_i}(i) \rangle \right] \right\}. \tag{A3}$$

We notice that every permutation π_α can be expressed uniquely as a product of cycles with no common elements. Moreover, the signature as well as the result from the integration over the positions and the summation over the spin states in (A3) is the same for all the permutations that have the same cyclic structure. The latter invariance is due to two reasons: if π_α and π'_α have the same cyclic structure, there exists a permutation σ_α such that $\pi'_\alpha = \sigma_\alpha^{-1} \pi_\alpha \sigma_\alpha$ and the Hamiltonian is unchanged under a permutation of the indices of the particles of the same species. The set of the permutations π_α of N_α elements can be seen as a partition in conjugate classes, each of which is made up with permutations with the same cyclic structure. Each class is characterized by a sequence $\{n_p^\alpha\}_{p=1,\ldots,N_\alpha}$, where n_p^α denotes the number of cycles with a size p in the decomposition of π_α into cycles that have no common element. The n_p^α's obey the constraint

$$N_\alpha = \sum_{p=1}^{N_\alpha} p n_p^\alpha . \tag{A4}$$

The signature of the permutation π_α only depends on the number of elements N_α and on the total number of independent cycles $\Sigma_{p=1}^{N_\alpha} n_p^\alpha$ and reads [43]

$$(-1)^{(N_\alpha - \Sigma_{p=1}^{N_\alpha} n_p^\alpha)} . \tag{A5}$$

The number of permutations in the same class is

$$\frac{N_\alpha!}{\prod_{p=1}^{N_\alpha} [n_p^\alpha! p^{n_p^\alpha}]} . \tag{A6}$$

[$N_\alpha!$ is the total number of permutations between N_α objects, $n_p^\alpha!$ is the number of global permutations between cycles with the same size p, and p is the number of (cyclic) permutations that act only on the objects involved in one of the n_p^α cycles with a length p and that do not change the order of the elements inside the cycle.] Equation (A3) can be written as

$$\Xi = \sum_{\{N_\alpha\}_{\alpha=1,\ldots,n_s}} \sideset{}{^*}\sum_{\{n_p^\alpha\}^{\alpha=1,\ldots,n_s}_{p=1,\ldots,N_\alpha}} \prod_\alpha \left[\frac{\eta_\alpha^{(N_\alpha - \Sigma_{p=1}^{N_\alpha} n_p^\alpha)} e^{\beta\mu_\alpha N_\alpha}}{N_\alpha!} \frac{N_\alpha!}{\prod_p [n_p^\alpha! p^{n_p^\alpha}]} \right] \times \left\{ \int \prod_i d\mathbf{r}_i \sum_{\{S^z_{\alpha_i}(i)\}_{i=1,\ldots,\Sigma_\alpha^{N_\alpha}}} \left[\bigotimes_i \langle \mathbf{r}_{\pi^0(i)}, S^z_{\alpha_i}(\pi^0(i)) | \right] e^{-\beta \hat{H}_{\{N_\alpha\}}} \left[\bigotimes_i | \mathbf{r}_i, S^z_{\alpha_i}(i) \rangle \right] \right\}. \tag{A7}$$

The notation Σ^* means that, for each α, the n_p^α must obey the constraint (A4). η_α is equal to 1 for the bosons and -1 for the fermions and π^0 is a composition of some particular permutations π_α^0 in the class of the permutations specified by the sequence $\{n_p^\alpha\}_{p=1,\ldots,N_\alpha}$.

Since the Hamiltonian does not depend on the spin, the contribution from the spinorial part of the states factorizes out into the product of the contributions from the spin configurations of the particles that are permuted inside each cycle of the π_α^0's. For a given π^0, we replace the index i by a quadruplet (α,p,k,l) that labels a particle of species α, which is permuted under π^0 inside the kth cycle of length p and has an index l inside this cycle. Then the term in curly brackets in (A7) reads

$$\prod_\alpha \prod_p \prod_{k=1}^{n_p^\alpha} \left[\sum_{\{S_\alpha^z(p,k,l)\}_{l=1,\ldots,p}} \langle S_\alpha^z(\pi_\alpha^0(p,k,l)) | S_\alpha^z(p,k,l) \rangle \right] \times \int \prod_i d\mathbf{r}_i \langle \{\mathbf{r}_{\pi^0(i)}\} | e^{-\beta \hat{H}_{\{N_\alpha\}}} | \{\mathbf{r}_i\} \rangle, \tag{A8}$$

with $|\{\mathbf{r}_i\}\rangle \equiv \otimes_i |\mathbf{r}_i\rangle$. The contribution from a configuration $\{S_\alpha^z(p,k,l)\}_{l=1,\ldots,p}$ is different from zero only if all the particles that are permuted in the cycle have the same spin state $|S_\alpha^z\rangle$ in this configuration. Then the contribution of the configuration is equal to 1 for every value that S_α^z can take among the $2S_\alpha + 1$ possible ones. Subsequently, the contribution from the summation over the configurations

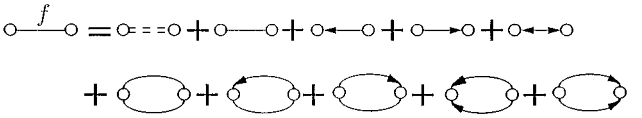

FIG. 6. Diagrammatic representation of the decomposition (B1) of an f bond into the ten auxiliary bonds $\tilde{f}'$ that are adequate for the resummation of the weights associated with the points in the prototype diagrams $\mathbb{P}$ of the loop-fugacity expansions. The truncated f_{TT} bond (B2) is denoted by a double dotted line. The extra bonds with respect to the decomposition of Fig. 2 are drawn as multiple bonds with the graphical conventions of Fig. 2. (The symmetry factor 2 does not appear in the representation of a diagram.)

$\{S_\alpha^z(p,k,l)\}_{l=1,\ldots,p}$ is $2S_\alpha+1$ for each cycle (α,p,k) and the total contribution from the spinorial part reads

$$\prod_{\alpha=1}^{n_s}\prod_{p=1}^{N_\alpha}(2S_\alpha+1)^{n_p^\alpha}. \tag{A9}$$

Eventually, by using (A4)–(A6) and (A9), we rewrite Ξ in such a way that N_α does not appear in the coefficients. Thus the constraint (A4), with $N_\alpha=1,\ldots,\infty$, can be released, n_α^p and p now run from 0 to ∞, and (A7) leads to the formula (3.1).

APPENDIX B

In this appendix we resum the Coulomb divergencies of the diagrams corresponding to the loop-fugacity expansion of the density of loops. The resummation scheme leads to formula (5.36) and the main differences with the loop-density expansion of the Ursell function, which arise from the existence of articulation points, are stressed at the beginning of Sec. V D.

In order to get finite weights $w(\mathscr{P})$ in (5.36), the resummation of the Coulomb rings must include the ring $\int d\mathscr{L}_j \frac{1}{2}[f^{cc}(\mathscr{P}_i,\mathscr{L}_j)]^2$, which itself gives a divergent contribution because of the nonintegrability of $(1/R)^2$. If we used the decomposition (5.6) of f to build the auxiliary diagrams $\tilde{\mathbb{G}}$, the above ring would miss in the resummation because two points in a $\tilde{\mathbb{G}}$ diagram can be linked by at most one $\tilde{f}$ bond. In order to make the above ring appear in the $\tilde{\mathbb{G}}$ diagrams together with the other rings with one internal Coulomb point, namely, $\int d\mathscr{L}_j f^{cc}(\mathscr{P}_i,\mathscr{L}_j)f^{mc}(\mathscr{P}_i,\mathscr{L}_j)$ and $\int d\mathscr{L}_j \frac{1}{2}[f^{mc}(\mathscr{P}_i,\mathscr{L}_j)]^2$, f is written as the sum of the following $\tilde{f}'$-bonds:

$$f^{cc}(c_i,c_j),\ \ f^{mc}(\mathscr{L}_i,c_j),\ \ f^{cm}(c_i,\mathscr{L}_j),\ \ f^{mm}(\mathscr{L}_i,\mathscr{L}_j),$$
$$f_{TT}(\mathscr{L}_i,\mathscr{L}_j),\ \ \tfrac{1}{2}[f^{cc}(c_i,c_j)]^2,\ \ f^{cc}(c_i,c_j)f^{mc}(\mathscr{L}_i,c_j),$$
$$f^{cc}(c_i,c_j)f^{cm}(c_i,\mathscr{L}_j),\ \ \tfrac{1}{2}[f^{mc}(\mathscr{L}_i,c_j)]^2,\ \ \tfrac{1}{2}[f^{cm}(c_i,\mathscr{L}_j)]^2, \tag{B1}$$

where f_{TT} is a truncated form of f_T defined in (5.7),

$$\begin{aligned}f_{TT}(\mathscr{L}_i,\mathscr{L}_j)\equiv{}& f_T(\mathscr{L}_i,\mathscr{L}_j)-\tfrac{1}{2}[f^{cc}(c_i,c_j)]^2\\ &-f^{cc}(c_i,c_j)f^{mc}(\mathscr{L}_i,c_j)\\ &-f^{cc}(c_i,c_j)f^{cm}(c_i,\mathscr{L}_j)-\tfrac{1}{2}[f^{mc}(\mathscr{L}_i,c_j)]^2\\ &-\tfrac{1}{2}[f^{cm}(c_i,\mathscr{L}_j)]^2.\end{aligned} \tag{B2}$$

The diagrammatic representation of the decomposition (B1) is shown in Fig. 6 and a $\tilde{\mathbb{G}}$ diagram is drawn in Fig. 7(a).

After a resummation of the Coulomb chains defined as in Sec. V B, the $\mathbb{P}$ diagrams in formula (5.36) are defined univocally, if we introduce charge-charge, multipole-charge, and charge-multipole bonds F_z^{cc}, F_z^{mc}, and F_z^{cm}, respectively, that satisfy the following excluded-convolution rule: the convolutions $F_z^{cc}*F_z^{cc}$, $F_z^{mc}*F_z^{cc}$, $F_z^{cc}*F_z^{cm}$, and

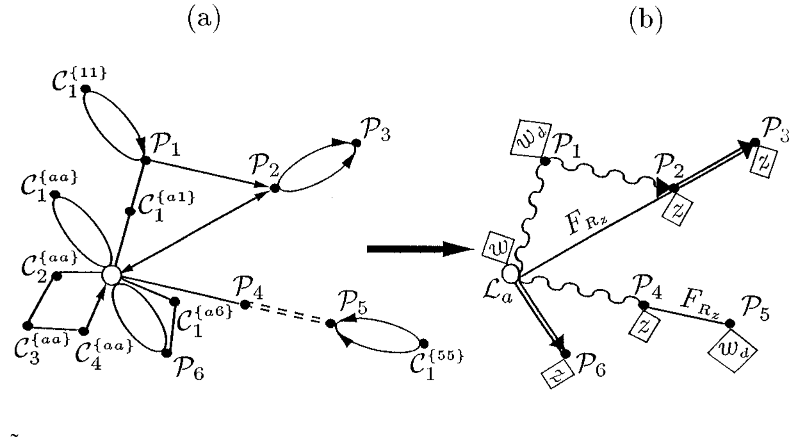

FIG. 7. Typical auxiliary $\tilde{\mathbb{G}}$ diagram and the prototype diagram $\mathbb{P}$ to which it contributes by the resummation process defined in Sec. V D. An F_{R_z} bond is denoted by a solid line with a superscript F_{R_z}, while the nonsymmetric truncated bond $F_{R_zT}^{cm}$ is denoted by a double solid line with an arrow pointing to the end point that is not linked to any other point. w_d is a short notation for the weight w_{dressed} and z is the value of the weight w_{bare}. $\mathscr{C}_1^{\{a1\}}$ is a bare point that disappears in the resummation process, whereas $\mathscr{P}_1$ is a dressed point involved in a convolution f^{cc} $(\mathscr{C}_1^{\{a1\}},\mathscr{P}_1)*f^{cm}(\mathscr{P}_1,\mathscr{P}_2)$; $\mathscr{P}_1$ is left over in the resummation process and appears as the intermediate point of a convolution $F^{cc}(\mathscr{L}_a,\mathscr{P}_1)*F^{cm}(\mathscr{P}_1,\mathscr{P}_2)$ in the corresponding $\mathbb{P}$ diagram. $\mathscr{C}_1^{\{55\}}$ is a Coulomb point and the bond $\frac{1}{2}[f^{cm}]^2(\mathscr{C}_1^{\{5,5\}},\mathscr{P}_5)]^2$ contributes to the weight of $\mathscr{P}_5$, whereas $\mathscr{P}_3$ is not a Coulomb point, and the bond $\frac{1}{2}[f^{cm}]^2(\mathscr{P}_2,\mathscr{P}_3)$ contributes to the truncated dressed bond $F_{R_zT}^{cm}(\mathscr{P}_2,\mathscr{P}_3)$, in which the contributions such as $\frac{1}{2}[f^{mc}(\mathscr{P}_2,\mathscr{P}_3)]^2$ are substracted (because $\mathscr{P}_3$ is linked only to $\mathscr{P}_2$). $\mathscr{P}_5$, which is linked only to $\mathscr{P}_4$, is a dressed point, and the bond linking $\mathscr{P}_4$ to $\mathscr{P}_5$ is a full bond $F_{R_z}(\mathscr{P}_4,\mathscr{P}_5)$, whereas $\mathscr{P}_6$, which is linked only to $\mathscr{L}_a$, is a bare point and the bond between $\mathscr{L}_a$ and $\mathscr{P}_6$ is a truncated bond $F_{R_zT}^{cm}(\mathscr{L}_a,\mathscr{P}_6)$.

$F_z^{mc}*F_z^{cm}$ appear only if the intermediate point of the chain has Coulomb rings attached to it in the corresponding $\tilde{\mathbb{G}}$ diagrams. [The weight $w(\mathscr{P}_i)$ does not depend on the position $\mathbf{R}_i$ of the point $\mathscr{P}_i$.] Thus it is convenient to introduce the notion of ''bare'' and ''dressed'' points, according to whether or not the point carries Coulomb rings in the $\tilde{\mathbb{G}}$ diagrams that lead to the considered diagram $\mathbb{P}$. In Fig. 7(a), $\mathscr{C}_1^{\{a1\}}$ is a bare point, whereas $\mathscr{P}_1$ is a dressed point. The excluded-convolution rule can be expressed as follows: there cannot be convolutions $F_z^{cc}*F_z^{cc}$, $F_z^{mc}*F_z^{cc}$, $F_z^{cc}*F_z^{cm}$, or $F_z^{mc}*F_z^{cm}$ where the intermediate point would be a bare point. Moreover, in order not to count twice the Coulomb rings with at least one intermediate point that already appear in the weight of the dressed points that are also articulation points, we must define two kinds of dressed bonds: a ''truncated'' dressed bond $F_{R_zT}^{cm}(\mathscr{P}_i,\mathscr{P}_j)$, if $\mathscr{P}_j$ is a bare point that is linked only to $\mathscr{P}_i$ in the $\mathbb{P}$ diagram [see the resummed bonds linking respectively $\mathscr{P}_2$ to $\mathscr{P}_3$ and $\mathscr{L}_a$ to $\mathscr{P}_6$ in Fig. 7(b)]; a ''full'' dressed bond F_{R_z} in other cases [see the bond linking $\mathscr{L}_a$ to $\mathscr{P}_2$ and that linking $\mathscr{P}_4$ to $\mathscr{P}_5$ in Fig. 7(b)]. The nonsymmetric truncated dressed bond is equal to the full dressed bond minus the contributions from the Coulomb rings with at least one intermediate Coulomb point (where $\mathscr{P}_j$ is an intermediate Coulomb point)

$$F_{R_zT}^{cm}=F_{R_z}-\tfrac{1}{2}[F_z^{cc}(c_i,c_j)]^2-F_z^{cc}(c_i,c_j)F_z^{mc}(\mathscr{L}_i,c_j)-\tfrac{1}{2}[F_z^{mc}(\mathscr{L}_i,c_j)]^2. \qquad \text{(B3)}$$

The calculation of the resummed bonds is analogous to that of Sec. V C because the bonds f^{cc}, f^{mc}, and f^{cm} play the same role in both processes and the role played by f^{mm} and f_T in Sec. V C is performed in Sec. V D by f^{mm}, f_{TT}, $[f^{cc}]^2/2$, $f^{cc}f^{mc}$, $f^{cc}f^{cm}$, $[f^{mc}]^2/2$, and $[f^{cm}]^2/2$, with the relation (B2). The bonds F_z^{cc}, F_z^{mc}, F_z^{cm}, and F_{R_z} have the same expressions as the corresponding resummed bonds in the loop-density expansion of the Ursell function, except that the square inverse length κ^2 is replaced by

$$\kappa_z^2=4\pi\beta\sum_{\alpha=1}^{n_s}e_\alpha^2\sum_{p=1}^{\infty}p^2\int D(\mathbf{X})z_{\alpha,p}(\mathbf{X})=\frac{\partial}{\partial\mu_\alpha}\left(4\pi\sum_{\alpha=1}^{n_s}e_\alpha^2\sum_{p=1}^{\infty}p\int D(\mathbf{X})z_{\alpha,p}(\mathbf{X})\right)\Bigg|_\beta. \qquad \text{(B4)}$$

Indeed, the weight of the points in the loop-fugacity expansion of the loop density is $z(\mathscr{L})$ instead of $\rho(\mathscr{L})$ and since $z(\mathscr{L})$ depends on μ_α only through the term $[\exp(\beta\mu_\alpha)]^p$, the relation $\beta pz(\mathscr{L})=\partial z(\mathscr{L})/\partial\mu_\alpha)$ is valid for an interacting system as well as for an ideal gas. In the case of bosons $z(\mathscr{L})>0$, κ_z^2 is positive and κ_z is real. Moreover, according to (3.16), κ_z^2 is bounded by the finite value κ_{RPA}^2,

$$0<\kappa_z^2<4\pi\beta\sum_{\alpha=1}^{n_s}e_\alpha^2\sum_{p=1}^{\infty}p^2\int D(\mathbf{X})z_{\alpha,p}^0(\mathbf{X};\beta,\mu_\alpha)=4\pi\sum_{\alpha=1}^{n_s}e_\alpha^2\left.\frac{\partial\rho_\alpha^{0Q}(\beta,\mu_\alpha)}{\partial\mu_\alpha}\right|_\beta=\kappa_{\text{RPA}}^2. \qquad \text{(B5)}$$

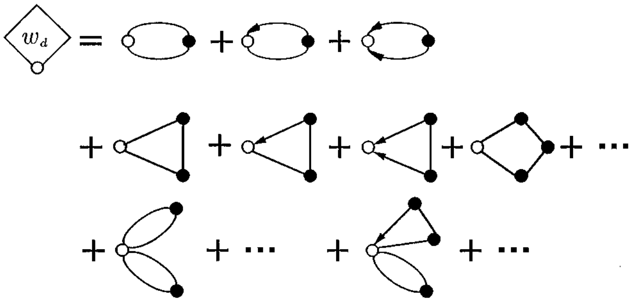

FIG. 8. Diagrammatic representation of the weight $w(\mathscr{P})$ (B8) of a dressed point. An arbitrary number of Coulomb rings are attached to $\mathscr{P}$ in the $\tilde{\mathbb{G}}$ diagram and the rings contain an arbitrary number of internal points.

In the case of fermions, the series (B4) over p is alternate. Let us consider a noninteracting gas of quantum particles with mass m_α and spin $2S_\alpha+1$ (with $\alpha=1,\ldots,n_s$) at inverse temperature β and with a chemical potential μ_α^{*0}, such that its density satisfies $\rho_\alpha^{0Q}(\beta,\mu_\alpha^{*0})=\Sigma_{p=1}^{\infty}p\int D(\mathbf{X})z_{\alpha,p}(\mathbf{X};\beta,\mu_\alpha)$. Then

$$\kappa_z^2=4\pi\beta\sum_{\alpha=1}^{n_s}e_\alpha^2\left.\frac{\partial\rho_\alpha^{0Q}(\beta,\mu_\alpha^{*0})}{\partial(\beta\mu_\alpha)}\right|_\beta. \qquad \text{(B6)}$$

This formula is slightly different from the RPA expression (6.13). At this point, we may only conjecture that μ_α^{*0} is an increasing function of μ_α with a finite derivative, so that Eq. (B6) implies that κ_z^2 is positive and finite also in the case of fermions.

The weight of a bare point is merely

$$w_{\text{bare}}(\mathscr{P})=z(\mathscr{P}). \qquad \text{(B7)}$$

The weight of a dressed point $w_{\text{dressed}}(\mathscr{P})$ is the sum of the contributions from all the Coulomb rings that may be attached directly to the point $\mathscr{P}$ in the $\tilde{\mathbb{G}}$ diagrams (see Fig. 8). According to the same topological argument as that used to reexpress F_R in terms of exponentials of sums of single chains,

$$w_{\text{dressed}}(\mathscr{P})=z(\mathscr{P})[e^{I_r(\mathscr{P})}-1], \qquad \text{(B8)}$$

where $I_r(\mathscr{P})$ is the sum of all the Coulomb rings with at least one internal point, in which the two bonds attached to $\mathscr{P}$ are either two f^{cc} bonds, one f^{cc} and one f^{mc} bond, or two f^{mc} bonds. The root point $\mathscr{L}_a$ has the particular weight

$$w(\mathscr{L}_a)=z(\mathscr{L}_a)e^{I_r(\mathscr{L}_a)}. \qquad \text{(B9)}$$

According to the definition (5.21) of ψ_{chain}, $I_r(\mathscr{P})=-\frac{1}{2}\beta e_\alpha^2\psi_{\text{chain}}(\mathscr{P},\mathscr{P})$, where 2 is the symmetry factor of the sum of rings $\psi_{\text{chain}}(\mathscr{P},\mathscr{P})$. $I_r(\mathscr{P})$ has a finite value given by (5.29),

$$I_r(\mathscr{P})=\frac{1}{2}\,\beta e_\alpha^2\int_0^p d\tau\int_0^p d\tau'\;\frac{1-e^{-\kappa_z|\mathbf{X}(\tau)-\mathbf{X}(\tau')|}}{|\mathbf{X}(\tau)-\mathbf{X}(\tau')|}. \tag{B10}$$

The inequality $[(1-e^{-\kappa_z x})/x]<\kappa_z$, for $x>0$, can be used to find upper bounds independent from the internal variables $\mathbf{X}$. Since the internal energy $E_\beta^{\text{int}}(\mathscr{P})$ (3.7) of a loop is positive, $-\beta E_\beta^{\text{int}}(\mathscr{P})+I_r(\mathscr{P})<(\beta e_\alpha^2/2)p^2\kappa_z$. However, a better upper bound can be found if we exhibit the self-interaction energy $E_{\text{self}}^{\phi_z}(\mathscr{P})$ of a loop with the potential $\phi_z(\mathbf{r})=\exp(-\kappa_z r)/r$ instead of $v_C(\mathbf{r})$. $E_{\text{self}}^{\phi_z}(\mathscr{P})$ is given by (3.15) with $\exp[-\kappa_z|\mathbf{X}(\tau)-\mathbf{X}(\tau')|$ in place of $1/|\mathbf{X}(\tau)-\mathbf{X}(\tau')|$ and it contains neither short-distance nor large-distance singularities. According to (3.15) and (B10),

$$\begin{aligned}-\beta E_{\text{self}}(\mathscr{P})+I_r(\mathscr{P})={}&-\beta E_{\text{self}}^{\phi_z}(\mathscr{P})+\frac{1}{2}\,\beta e_\alpha^2\int_0^p d\tau\int_0^p d\tau'\\&\times\{1-(1-\delta_{P(\tau),P(\tau')})\,\delta([\tau-P(\tau)]\\&-[\tau'-P(\tau')])\}\;\frac{1-e^{-\kappa_z|\mathbf{X}(\tau)-\mathbf{X}(\tau')|}}{|\mathbf{X}(\tau)-\mathbf{X}(\tau')|}.\end{aligned} \tag{B11}$$

This rewritting makes no spurious divergencies appear. The first term on the right-hand side of (B11) is negative and the second term is lower than

$$\begin{aligned}\tfrac{1}{2}\beta e_\alpha^2\kappa_z\int_0^p d\tau\int_0^p d\tau'\{1-(1-\delta_{P(\tau),P(\tau')})\,\delta([\tau-P(\tau)]\\-[\tau'-P(\tau')])\}=p\,\tfrac{1}{2}\beta e_\alpha^2\kappa_z.\end{aligned} \tag{B12}$$

Eventually, we get

$$\begin{aligned}|z(\mathscr{P})|e^{I_r(\mathscr{P})}&=|z^*_{\alpha,p}(\mu_\alpha)|e^{[-\beta E_\beta^{\text{int}}(\mathscr{P})+I_r(\mathscr{P})]}\\&<|z^*_{\alpha,p}(\mu_\alpha+\tfrac{1}{2}e_\alpha^2\kappa_z)|e^{-E_\beta^0(\{\mathbf{t}_l\})},\end{aligned} \tag{B13}$$

The upper bound in (B13) is the fugacity of a noninteracting loop in which the particles have a chemical potential that includes the self-energy of a point particle creating a potential in the manner of Debye, $\exp(-\kappa_z r)/r$. While $|w_{\text{bare}}(\mathscr{P})|$ (B7) is bounded by the absolute value of the free-loop density $|z^*_{\alpha,p}(\mu_\alpha)|\exp[-E_\beta^0(\{\mathbf{t}_l\})]$, according to (3.16), (B13) provides an upper bound for $|w_{\text{dressed}}(\mathscr{P})|\leqslant|w_{\text{bare}}(\mathscr{P})|+|z(\mathscr{P})|\exp[I_r(\mathscr{P})]$, according to (B8).

APPENDIX C

In this appendix we study the integrability of the resummed prototype diagrams for both the loop-fugacity expansion of the density of loops and the loop-density expansion of the Ursell function between loops. The dependence of the density of loops upon the extension of loops is also discussed.

Let us first consider the convergence of the integral $I_{\mathbb{P}}$ associated with the diagram $\mathbb{P}$,

$$I_{\mathbb{P}}(\mathscr{L}_a)\equiv\frac{1}{S_{\mathbb{P}}}\int\prod_{m=1}^{M}[d\mathscr{P}_m w(\mathscr{P}_m)]\Big[\prod F_z\Big]_{\mathbb{P}}. \tag{C1}$$

At short distances in the space of the loop variables, i.e., when a curve $\boldsymbol{\Omega}_i$ nearly coincides with a curve $\boldsymbol{\Omega}_j$, every bond F_z proves to be integrable. Indeed, the singularities $1/|\mathbf{R}_i-\mathbf{R}_j|$ and $1/|\boldsymbol{\Omega}_i(\tau)-\mathbf{R}_j|$ in the various bonds are integrable at short distances, as well as the singularities $1/|\mathbf{R}_i-\mathbf{R}_j|^2$, $1/[|\mathbf{R}_i-\mathbf{R}_j||\boldsymbol{\Omega}_i(\tau)-\mathbf{R}_j|]$, and $1/|\boldsymbol{\Omega}_i(\tau)-\mathbf{R}_j|^2$ in $F_{R_zT}^{cm}$. Moreover, when $\boldsymbol{\Omega}_i$ tends to $\boldsymbol{\Omega}_j$, $\phi_{\text{elect}}(\mathscr{P}_i,\mathscr{P}_j)-v_{\text{elect}}(\mathscr{P}_i,\mathscr{P}_j)$ tends to a constant $-p_ip_j\kappa_z$, while the divergence of $\exp[-\beta_{ij}v(\mathscr{P}_i,\mathscr{P}_j)]$ for $e_{\alpha_i}e_{\alpha_j}<0$ is smoothed out by the functional integration over the Brownian paths $\boldsymbol{\xi}_l^i$ contained in $D(\mathbf{X}_i)$. (The space of the paths that cross one another has a zero measure.) Henceforth, the five bonds F_z are integrable at short distances separately. The divergences that might come from products of bonds F_z are in fact spurious since they are introduced by the decomposition of the bond f into auxiliary bonds $\tilde{f}$, whereas products of bonds f are integrable at short distances. By suitably collecting the $\mathbb{P}$ diagrams together, the possible short-distance divergencies of $[\Pi F_z]_{\mathbb{P}}$ must disappear, as is the case in the classical prototype diagrams introduced by Meeron [40,41].

The integrability of $I_{\mathbb{P}}(\mathscr{L}_a)$ at large distances in the space of the loop variables reduces to the study of the large-R_{ij} behavior of the bonds $F_z(\mathscr{P}_i,\mathscr{P}_j)=F_z(\mathbf{R}_{ij},\mathbf{X}_i,\mathbf{X}_j)$ between internal points and to the study of the large-$|\boldsymbol{\Omega}_a(\tau)-\mathbf{R}_j|$ behavior of the bonds $F_z(\mathscr{L}_a,\mathscr{P}_j)=F_z(\boldsymbol{\Omega}_a,\mathbf{R}_j,\mathbf{X}_j)$ between the root point $\mathscr{L}_a$ and internal points $\mathscr{P}_j$. Indeed, the integrals over the variables $\mathbf{X}=(\{\mathbf{t}_l\}_{l=1,\dots,p-1},\{\boldsymbol{\xi}_l\}_{l=1,\dots,p})$ of any polynomial in the variables $\mathbf{X}$ multiplied by $w(\mathscr{P})=w_{\alpha,p}(\mathbf{X})$ are finite because, according to the conclusion of Appendix B, the weight $w(\mathscr{P})$ is bounded by the Gaussian $\exp[-(\frac{1}{2}\Sigma_l\,t_l^2)/\lambda^2]$, while the measure $\Pi_{l=1}^p D(\boldsymbol{\xi}_l)$ is also Gaussian. At large distances, the dressed bonds F_z^{cc}, F_z^{cm}, and F_z^{mc} have exponential falloffs, while F_{R_z} and $F_{R_zT}^{cm}$ decay algebraically as the leading term in the asymptotic behavior of $v-v_{\text{elect}}$. For a dressed bond between two internal points $\mathscr{P}_i$ and $\mathscr{P}_j$, this tail starts as the $1/R^3$ term $W_3(\mathbf{R},\mathbf{X}_i,\mathbf{X}_j)$ given in (5.34). For a dressed bond between $\mathscr{L}_a$ and an internal point $\mathscr{P}_j$, $v-v_{\text{elect}}$ behaves as

$$\begin{aligned}&\int_0^{p_a}d\tau\int_0^{p_j}d\tau'\{\delta([\tau-P(\tau)]-[\tau'-P(\tau')])-1\}\\&\quad\times\left\{\mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_{\mathbf{R}_j}\left(\frac{1}{|\boldsymbol{\Omega}_a(\tau)-\mathbf{R}_j|}\right)\right.\\&\quad+\frac{1}{2}\,[\mathbf{X}_j(\tau')\cdot\boldsymbol{\nabla}_{\mathbf{R}_j}]^2\left(\frac{1}{|\boldsymbol{\Omega}_a(\tau)-\mathbf{R}_j|}\right)\\&\quad\left.+O\left(\frac{1}{|\boldsymbol{\Omega}_a(\tau)-\mathbf{R}_j|^4}\right)\right\}.\end{aligned} \tag{C2}$$

The existence of articulation points implies that, when two clusters of points in a $\mathbb{P}$ diagram are separated by a distance R, each cluster keeping a bounded size, the integrand $[\Pi F_z]_{\mathbb{P}}$ may decay as $v-v_{\text{elect}}$, if the two clusters are linked to each other by only one dressed bond. The integrals associated with the corresponding $\mathbb{P}$ diagrams can be only conditionally convergent.

An analogous problem exists for the coefficients in the fugacity expansions for the pressure and correlations in two-dimensional classical plasmas without any resummation. In the case of neutral systems of particles with arbitrary charges (non-charge-symmetric case) and hard cores, Speer [39] has shown that the coefficient of order N in the fugacity z does exist for sufficiently low temperatures $T \leqslant T_N$ if it is calculated as follows. The configuration space is subdivided into regions corresponding to spatial configurations in which the particles are grouped hierarchically into neutral clusters, such that the interparticle distances within a cluster are an order of magnitude less than the distance from the considered cluster to any other disjoint cluster. Because of the existence of the boundary, a further subdivision of configuration space into subregions must be made. Within a given subregion, each cluster may be either averaged over orientations or confined to lie near the boundary of the system of finite volume Λ. At sufficiently low temperatures, when calculating the contribution from the various configurations to the coefficient of z^N, the contribution from any subregion involving a cluster confined near the boundary proves to vanish in the infinite volume limit, while the contribution from subregions where all the configurations are in the bulk is finite, if the orientations of the neutral clusters are averaged before the integration over the remaining coordinates. In the present case, any possible $1/R^2$ or $1/R^3$ terms in the large-$|\mathbf{\Omega}_a - \mathbf{R}_j|$ or large-R_{ij} expansion of $[\Pi \mathrm{F}_z]_{\mathbb{P}}$ is generated by the existence of an F_{R_z} bond or a $F^{cm}_{R_z T}$ bond that links two sets of loops that are not connected to each other by any other bond. By analogy with the result of Ref. [39], we expect that the integration over the internal degrees of freedom $\mathbf{X}$ (shapes of the loops) and the relative positions of the loops inside each cluster must be performed before the integration over the vector $\mathbf{R}$ that characterizes the relative position of the two clusters. Let $\mathscr{P}_0$ be the end point of the bond that is in the same cluster as $\mathscr{L}_a$ and $\mathscr{P}_i$ the other one. ($\mathscr{P}_0$ may coincide with $\mathscr{L}_a$.) We use the notations $\mathbf{R}_{0a} = \mathbf{R}_a - \mathbf{R}_0$ and $\mathbf{R}_{0j} = \mathbf{R}_j - \mathbf{R}_0$. After integration over the loops inside the clusters, except $\mathscr{L}_a$, $\mathscr{P}_0$, and $\mathscr{P}_j$, and integration over the shape $\mathbf{X}_j$ of $\mathscr{P}_j$, the possible nonintegrable tail that may come from the leading term of $v - v_{\mathrm{elect}}$ in the asymptotic behavior of $\int D(\mathbf{X}_j) w(\mathscr{P}_j) \int [\Pi_{m \neq 0,j} d\mathscr{P}_m w(\mathscr{P}_m)][\Pi \mathrm{F}_z]_{\mathbb{P}}$ has the following form. If $\mathscr{P}_0$ does not coincide with $\mathscr{L}_a$, the latter leading term is given by $W_3(\mathbf{R}_{0j}, \mathbf{X}_0, \mathbf{X}_j)$, which reads

$$\beta_{ij} \int_0^{p_0} d\tau \int_0^{p_j} d\tau' \{\delta([\tau - P(\tau)] - [\tau' - P(\tau')]) - 1\}$$

$$\times G_1(\mathbf{R}_{a0}, \mathbf{X}_a, \mathbf{X}_0) \int D(\mathbf{X}_j) G_2(|\mathbf{X}_j|)$$

$$\times [\mathbf{X}_0(\tau) \cdot \boldsymbol{\nabla}_{\mathbf{R}_{0j}}][\mathbf{X}_j(\tau') \cdot \boldsymbol{\nabla}_{\mathbf{R}_{0j}}] \left(\frac{1}{R_{0j}} \right); \tag{C3}$$

If $\mathscr{P}_0$ coincides with $\mathscr{L}_a$, the algebraic tail is given by the asymptotic behavior of $[v - v_{\mathrm{elect}}](\mathbf{\Omega}_a, \mathbf{R}_j, \mathbf{X}_j)$ [see (C2)], with the result

$$-\beta_{ij} \int_0^{p_0} d\tau \int_0^{p_j} d\tau' \{\delta([\tau - P(\tau)] - [\tau' - P(\tau')]) - 1\}$$

$$\times \int D(\mathbf{X}_j) G_2(|\mathbf{X}_j|) \Bigg\{ \mathbf{X}_j(\tau') \cdot \boldsymbol{\nabla}_{\mathbf{R}_j} \left(\frac{1}{|\mathbf{\Omega}_a(\tau) - \mathbf{R}_j|} \right)$$

$$+ \frac{1}{2} [\mathbf{X}_j(\tau') \cdot \boldsymbol{\nabla}_{\mathbf{R}_j})]^2 \left(\frac{1}{|\mathbf{\Omega}_a(\tau) - \mathbf{R}_j|} \right)$$

$$+ O\left(\frac{1}{|\mathbf{\Omega}_a(\tau) - \mathbf{R}_j|^4} \right) \Bigg\}. \tag{C4}$$

In (C3) and (C4), we have omitted the dependences upon the variables α, p, and τ in the notation G_j. According to the rotational invariance of the various measures and bonds, the quantity $G_2(|\mathbf{X}_j|)$ is invariant under rotations of $\mathbf{X}_j$. As a consequence, after integration over $\mathbf{X}_j$, the term (C3) does vanish, as well as the first term in (C4), while the second term in (C4) is proportional to $\Delta[1/|\mathbf{\Omega}_a(\tau) - \mathbf{R}_j|]$ and is in fact short ranged. Eventually every graph $\mathbb{P}$ corresponds to a finite integral $I_{\mathbb{P}}(\mathscr{L}_a)$.

Now, we turn to the dependence of $\rho(\mathscr{L}_a) = \rho_{\alpha_a, p_a}(\mathbf{X}_a)$ on the variables $\mathbf{X}_a$. First $I_{\mathbb{P}}(\mathscr{L}_a)$ remains finite when one of the variables $(\{\mathbf{t}_l^a\}_{l=1,\ldots,p-1}, \{\boldsymbol{\xi}_l^a\}_{l=1,\ldots,p})$ becomes very large, i.e., when the extent of the curve $\mathbf{\Omega}_a$ becomes very large. Indeed, $I_{\mathbb{P}}(\mathscr{L}_a)$ defined in (C1) can be written as

$$I_{\mathbb{P}}(\mathscr{L}_a) = \frac{1}{S_{\mathbb{P}}} \int \prod_{j=1}^{J(\mathbb{P})} [d\mathscr{P}_j^* w(\mathscr{P}_j^*)]$$

$$\times \left[\prod_{j=1}^{J(\mathbb{P})} F(\mathscr{L}_a, \mathscr{P}_j^*) \right] g(\{\mathscr{P}_j^*\}_{j=1,\ldots,J(\mathbb{P})}), \tag{C5}$$

where the $\mathscr{P}_j^*$'s are the $J(\mathbb{P})$ points that are linked to $\mathscr{L}_a$ by an F_z bond in the graph $\mathbb{P}$. The dependence of $I_{\mathbb{P}}(\mathscr{L}_a) = I_{\mathbb{P}}(\mathbf{X}_a; \alpha_a, p_a)$ on the variable $\mathbf{X}_a$ originates from the bonds $F^{mc}(\mathscr{L}_a, \mathscr{P}_j^*)$, $F_{R_z}(\mathscr{L}_a, \mathscr{P}_j^*)$, or $F^{cm}_{R_z T}(\mathscr{L}_a, \mathscr{P}_j^*)$. [The bonds $F^{cc}(\mathscr{L}_a, \mathscr{P}_j^*)$ or $F^{cm}(\mathscr{L}_a, \mathscr{P}_j^*)$ depend on $\mathbf{R}_a$, but not on $\mathbf{X}_a$.] On one hand, $F^{mc}(\mathscr{L}_a, \mathscr{P}_j^*)$ is proportional to the potential in the manner of Debye created in $\mathbf{R}_j^*$ by a closed curve $\mathbf{\Omega}_a(\tau)$ with a density $\sigma_a(\mathbf{r}) = \int_0^{p_a} d\tau\ \delta(\mathbf{\Omega}_a(\tau) - \mathbf{r}]$. Thus, when the extent of the curve $\mathbf{\Omega}_a$ becomes very large, after integration over the internal points $\mathscr{P}_j^*$, the various bonds $F^{mc}(\mathscr{L}_a, \mathscr{P}_j^*)$ give a finite contribution to $I_{\mathbb{P}}(\mathscr{L}_a)$. On the other hand, when the distance between $\mathscr{P}_j^*$ and each point of the curve $\mathbf{\Omega}_a$ becomes large, the bonds $F_{R_z}(\mathscr{L}_a, \mathscr{P}_j^*)$ and $F^{cm}_{R_z T}(\mathscr{L}_a, \mathscr{P}_j^*)$ decay as the potential (C2), which is proportional to

$$\int d\mathbf{r} \int_0^1 ds [\tilde{\sigma}_a(\mathbf{r}, s) - \sigma_a(\mathbf{r})]$$

$$\times \sum_{l'=0}^{p_j - 1} \left\{ \mathbf{X}_j(l' + s) \cdot \boldsymbol{\nabla}_{\mathbf{R}_j} \left(\frac{1}{|\mathbf{r} - \mathbf{R}_j|} \right) + O\left(\frac{1}{|\mathbf{r} - \mathbf{R}_j|^3} \right) \right\}. \tag{C6}$$

The density $\tilde{\sigma}_a$ is defined as $\tilde{\sigma}_a(\mathbf{r}, s) = \int_0^{p_a} d\tau\ \delta(\tau - P(\tau) - s) \delta(\mathbf{\Omega}_a(\tau) - \mathbf{r}) = \sum_{l=0}^{p_a - 1} \delta(\mathbf{\Omega}_a(l + s) - \mathbf{r})$. The contribution of the potential (C6) to $I_{\mathbb{P}}(\mathbf{X}_a; \alpha_a, p_a)$ remains finite when the extent of the curve $\mathbf{\Omega}_a$ becomes infinite because $\sigma_a(\mathbf{r})$ and $\widetilde{\sigma}_a(\mathbf{r},s)$ are nonzero only on the curve $\mathbf{\Omega}_a$, which has essentially the space dimension of a Brownian trajectory. Since the measure $D(\boldsymbol{\xi}_a^l)$ is Gaussian, $\int[\Pi_{l=1}^{p} D(\boldsymbol{\xi}_a^l)] I_{\mathbb{P}}(\mathscr{L}_a)$ also remains finite when the size of a given $\mathbf{t}_l^a$ goes to infinity. At this point, we may only conjecture that the same is true for $[\Pi_{l=1}^{p} D(\boldsymbol{\xi}_a^l)] |\Sigma_{\mathbb{P}} I_{\mathbb{P}}|$. By using $\rho(\mathscr{L}_a) = w(\mathscr{L}_a) \Sigma_{\mathbb{P}} I_{\mathbb{P}}(\mathscr{L}_a)$ [see (5.36) and (C1)] and the bound (B13) over the weight w, we get

$$\left| \left[\prod_{l=1}^{p} D(\boldsymbol{\xi}_l^a) \right] \rho(\mathscr{L}_a) \right| < \left| z_{\alpha,p}^{*} (\mu_\alpha + \tfrac{1}{2} e_\alpha^2 \kappa_z) \right| \times \exp\left(-\frac{1}{2\lambda_\alpha^2} \sum_l [\mathbf{t}_l^a]^2 \right) \left[\prod_{l=1}^{p} D(\boldsymbol{\xi}_l^a) \right] \left| \sum_{\mathbb{P}} I_{\mathbb{P}}(\mathscr{L}_a) \right| . \tag{C7}$$

Thus $\rho(\mathscr{L}_a)$ and $\int[\Pi_{l=1}^{p} D(\boldsymbol{\xi}_a^l)] \rho(\mathscr{L}_a)$ are expected to decay faster than any inverse powerlaw in the variables $(\{\mathbf{t}_l^a\}_{l=1,\dots,p-1}, \{\boldsymbol{\xi}_l^a\}_{l=1,\dots,p})$ and $\{\mathbf{t}_l^a\}_{l=1,\dots,p-1}$, respectively. This is coherent with the fact that, in the present formalism, the density of quantum particles ρ_α^Q is deduced from $\rho(\mathscr{L}_a)$ by (4.3).

Let us consider the integrability of the Π diagrams when they are multiplied by $\rho(\mathscr{L}_a)\rho(\mathscr{L}_b)$ and integrated over $D(\mathbf{X}_a)D(\mathbf{X}_b)$. At short distances, the Π diagrams are integrable for the same reasons as the $\mathbb{P}$ diagrams. As argued previously, the density $\rho(\mathscr{P})$ is assumed to decrease very fast when the variables $(\{\mathbf{t}_l\}_{l=1,\dots,p-1}, \{\boldsymbol{\xi}_l\}_{l=1,\dots,p})$ become large and the integrability of the Π diagrams, when the distances between the loops become very large, is determined by the large-R behavior of the bonds F. These resummed bonds decay at least as the $1/R^3$ term W_3. Let us consider two sets of points in a given Π diagram: a set of internal points (loop variables) and the set containing the other internal points and the two root points. Since every Π diagram is connected and does not involve any articulation point, there are at least two paths of F bonds without any common intermediate point that join the former set to the latter one. When the distance R between the two clusters is far larger than the distances between the loops within each cluster, the integrand $[\Pi F]_\Pi$ corresponding to this configuration behaves at least as $(1/R^3)^2$ and the integral is absolutely convergent at large distances. The same mechanism ensures the integrability of the Mayer diagrams for classical dipolar fluids [44].

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